TOM V

TRIGONOMETRICAL SERIES

BY

ANTONI ZYGMUND

SECOND EDITION

CHELSEA PUBLISHING COMPANY
NEW YORK, N.Y.
1952

First Edition, Warsaw, 1935
Second Edition (Corrected Reprint of First Edition)
New York 1952

PRINTED IN THE UNITED STATES OF AMERICA

PREFACE.

The theory of trigonometrical series of a single variable is very extensive and is developing rapidly every year, but the space devoted to it in the existing text-books is small. There should, therefore, be room for a new book on this important subject.

The object of this treatise is to give an account of the present state of the theory; but, owing to the wide extent of the subject, it has been impossible to treat all parts in equal detail. In particular Fourier's integral, whose importance is more and more apparent, certainly deserves more space; but an adequate treatment would require a separate book.

Except for Lebesgue integration, an aquaintance with which is assumed, the book does not presuppose any special knowledge; the elements of Analysis are sufficient, except at one or two places. Besides the text, the book contains a number of miscellaneous examples and theorems, given at the end of every chapter. Some of these results are important; most of them are accompanied by indications of proofs, and so provide exercises for the reader.

Numbers in square brackets refer to the bibliography at the end of the book.

This book owes very much to Miss Mary L. Cartwright, D. Phil., of Girton College, and Dr. S. Saks of the University of Warsaw. Both kindly read the greater part of the manuscript and offered many valuable suggestions. Miss Cartwright has also helped me with the style in certain parts, and Dr. Saks in revising the proof-sheets. I wish to express my deep gratitude for the assistance they have given.

A. Zygmund.

Wilno, January 1935.

TABLE OF CONTENTS.

CHAPTER I.

Trigonometrical series and Fourier series.

1.1. Definitions. Trigonometrical series are series of the form

(1) $$\tfrac{1}{2} a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx),$$

where the coefficients $a_0, a_1, \dots, b_1, b_2, \dots$ are independent of the real variable x. It is convenient to provide the constant term of trigonometrical series with the factor 1/2. Except when otherwise stated, we shall suppose, always, that the coefficients of the trigonometrical series considered are real. Since all the terms of (1) are of period 2π, it is sufficient to study trigonometrical series in any interval of length 2π, e. g. in $(0,2\pi)$ or $(-\pi, \pi)$.

Consider the power series

(2) $$\tfrac{1}{2} a_0 + \sum_{k=1}^{\infty} (a_k - ib_k)\, z^k$$

on the unit circle: $z = e^{ix}$. The series (1) is the real part of (2).

The series

(3) $$\sum_{k=1}^{\infty} (a_k \sin kx - b_k \cos kx),$$

(with vanishing constant term) which multiplied by i and added to (1) gives the power series (2), is called *conjugate* to (1).

1.12. Summation of certain trigonometrical series. The fact that trigonometrical series are the real parts of power series facilitates in many cases finding the sums of the former. For example, the series

(1) $$P_r(x) = \tfrac{1}{2} + \sum_{k=1}^{\infty} r^k \cos kx, \quad Q_r(x) = \sum_{k=1}^{\infty} r^k \sin kx,$$

where $0 \leqslant r < 1$, are the real and imaginary parts of the series $\frac{1}{2} + z + z^2 + \ldots$, where $z = re^{ix}$, and so we obtain without difficulty

$$(2) \quad P_r(x) = \tfrac{1}{2} \cdot \frac{1 - r^2}{1 - 2r\cos x + r^2}, \quad Q_r(x) = \frac{r \sin x}{1 - 2r\cos x + r^2}.$$

Similarly, from the formula $\log 1/(1-z) = z + z^2/2 + \ldots$, we obtain

$$(3) \quad \begin{aligned} \sum_{k=1}^{\infty} \frac{\cos kx}{k} r^k &= \tfrac{1}{2} \log \frac{1}{1 - 2r\cos x + r^2}, \\ \sum_{k=1}^{\infty} \frac{\sin kx}{k} r^k &= \operatorname{arctg} \frac{r \sin x}{1 - r\cos x}, \end{aligned}$$

where $0 \leqslant r < 1$, $\operatorname{arctg} 0 = 0$. Denoting by $p_n(x)$, $q_n(x)$ the n-th partial sums ($n = 0, 1, 2, \ldots$) of the series (1) with $r = 1$, we obtain by the same argument

$$(4) \quad p_n(x) = \frac{\sin(n + \frac{1}{2})x}{2 \sin \frac{1}{2} x}, \quad q_n(x) = \frac{\cos \frac{1}{2} x - \cos(n + \frac{1}{2})x}{2 \sin \frac{1}{2} x}.$$

(A simple, although less natural, method of proving for example the first formula in (4) would be to multiply p_n by $2 \sin \frac{1}{2} x$ and to replace the products $\cos kx \cdot 2 \sin \frac{1}{2} x$ by differences of sines; then all the terms, except the last, cancel). From (4) we deduce that $p_n(x)$ and $q_n(x)$ are uniformly bounded, indeed less than $1/\sin \frac{1}{2}\varepsilon$ in absolute value, in every interval $0 < \varepsilon \leqslant x \leqslant 2\pi - \varepsilon$.

1.13. The complex form of trigonometrical series. Applying Euler's formulae to $\cos kx$, $\sin kx$, we may write the n-th partial sum of 1.1(1) in the form

$$s_n(x) = \tfrac{1}{2} a_0 + \tfrac{1}{2} \sum_{k=1}^{n} [(a_k - ib_k) e^{ikx} + (a_k + ib_k) e^{-ikx}].$$

If we define a_k, b_k for any integral k by the conditions $a_{-k} = a_k$, $b_{-k} = -b_k$, (thus, in particular, $b_0 = 0$), we see that s_n is the n-th symmetric partial sum, i. e. the sum of $2n+1$ terms with indices not exceeding n in absolute value, of the Laurent series

$$(1) \quad \sum_{k=-\infty}^{+\infty} c_k e^{ikx} \qquad (2c_k = a_k - ib_k).$$

Here c_{-k} is conjugate to c_k. Conversely, any series (1) with this property can be written in the form 1.1(1). Whenever we speak of convergence or summability of series (1), we shall always mean the limit, ordinary or generalized, of the symmetric partial sums.

The series conjugate to (1) may be obtained from the latter, replacing in it c_k by $-ic_k$ sign k, where sign $z = z/|z|$ if $z \neq 0$, and sign $0 = 0$.

1.2. Abel's transformation:

$$(1) \qquad \sum_{k=m}^{n} u_k v_k = \sum_{k=m}^{n-1} U_k (v_k - v_{k+1}) - U_{m-1} v_m + U_n v_n,$$

where $0 \leqslant m \leqslant n$, $U_k = u_0 + u_1 + \dots + u_k$ if $k \geqslant 0$, $U_{-1} = 0$. This formula, which can be easily verified, corresponds to integration by parts in the theory of integration, and is a very useful tool in the general theory of series. We shall call a sequence $v_0, v_1, \dots$ of *bounded variation* if the series $|v_0 - v_1| + |v_1 - v_2| + \dots$ is convergent. Without aiming at complete generality, we mention the following consequences of (1) in the case $m = 0$.

1.21 a) *If a series $u_0(x) + u_1(x) + \dots$ converges uniformly and $\{v_k\}$ is of bounded variation, the series $u_0(x)v_0 + u_1(x)v_1 + \dots$ converges uniformly.*

b) *If $u_0(x) + u_1(x) + \dots$ has its partial sums uniformly bounded, $\{v_k\}$ is of bounded variation and $v_k \to 0$, the series $u_0(x)v_0 + u_1(x)v_1 + \dots$ converges uniformly.*

1.22. A corollary of Abel's formula. If $v_m, v_{m+1}, \dots v_n$ are non-negative and non-increasing, the left-hand side of 1.2(1) does not exceed $2v_m \operatorname{Max} |U_k|$ $(m-1 \leqslant k \leqslant n)$ in absolute value. In fact, it does not exceed $\operatorname{Max} |U_k|$ multiplied by $(v_m - v_{m+1}) + \dots + (v_{n-1} - v_n) + v_m + v_n = 2v_m$.

1.23. Convergence of a class of trigonometrical series. The problems of convergence of 1.1(1) are, except in the trivial case when $|a_1| + |b_1| + |a_2| + |b_2| + \dots < \infty$, always delicate. Some rather special but, none the less, important results follow from Theorem 1.21. Applying it to the series

$$(1) \qquad \tfrac{1}{2} a_0 + \sum_{k=1}^{\infty} a_k \cos kx, \qquad \sum_{k=1}^{\infty} a_k \sin kx,$$

and taking into account the last remark in § 1.12, we obtain:

If $\{a_k\}$ is of bounded variation and $a_k \to 0$, in particular if a_k monotonically decreases to 0, the series (1) converge uniformly in any interval $0 < \varepsilon \leqslant x \leqslant 2\pi - \varepsilon$.

As regards the neighbourhood of $x = 0$, the behaviour of sine and cosine series may be quite different. In particular, the former always converge for $x = 0$, whereas the convergence of the latter is equivalent to that of $\frac{1}{2} a_0 + a_1 + \dots$ [1]).

Transforming the argument x we may present the last theorem in other, equivalent, forms. We shall be contented with the following statement.

If $\{a_k\}$ *is of bounded variation and* $a_k \to 0$, *then the series* $\frac{1}{2} a_0 - a_1 \cos x + a_2 \cos 2x - \dots$, $a_1 \sin x - a_2 \sin 2x + \dots$ *converge uniformly in* $(0, 2\pi)$, *except in arbitrarily small neighbourhoods of* $x = \pi$.

For the proof it is sufficient to replace in (1) x by $x + \pi$.

1.3. Orthogonal systems of functions. Fourier series. A system of real functions $\varphi_0(x), \varphi_1(x), \dots, \varphi_n(x), \dots$ defined in an interval (a, b) is said to be *orthogonal* in this interval if

$$(1) \qquad \int_0^1 \varphi_m(x)\, \varphi_n(x)\, dx \begin{array}{ll} = 0 & (m \neq n) \\ = \lambda_n > 0 & (m = n) \end{array} \qquad m, n = 0, 1, \dots$$

In particular, no φ_m vanishes identically. If $\lambda_0 = \lambda_1 = \dots = 1$, the system is said, in addition, to be *normal.* If $\{\varphi_n\}$ is orthogonal, $\{\varphi_n / \lambda_n^{1/2}\}$ is orthogonal and normal. The importance of orthogonal systems is based on the following fact. Suppose that a series $c_0 \varphi_0(x) + c_1 \varphi_1(x) + \dots$, where $c_0, c_1, \dots$ are constants, converges in (a, b) to a function $f(x)$. Multiplying both sides of the formula $f(x) = c_0 \varphi_0(x) + \dots + c_n \varphi_n(x) + \dots$ by $\varphi_n(x)$ and integrating over the range (a, b), we find, in virtue of (1), that

$$(2) \qquad c_n = \frac{1}{\lambda_n} \int_a^b f \varphi_n \, dx \quad (n = 0, 1, \dots).$$

This argument is purely formal, but in some cases, for example if the series defining f converges uniformly, φ_n are continuous and (a, b) is finite, it is easily justified. It suggests the following very important problem. Suppose that we have a function $f(x)$ defined in (a, b). Having formed the numbers c_n by means of (2), we write, quite formally,

$$(3) \qquad f(x) \sim c_0 \varphi_0(x) + c_1 \varphi_1(x) + \dots$$

[1]) See also Chapter V.

and call the series on the right the *Fourier series* of $f(x)$, with respect to the system $\{\varphi_n\}$. The numbers c_n are called the *Fourier coefficients* of f. The sign $\sim$ in (3) only means that the numbers c_n are connected with f by the formula (2) and does not imply in the least that the series is convergent, still less that it converges to f. Now, what are the properties of this series? In what sense does it 'represent' f?

This book is devoted to the study of one, very special but extremely important, orthogonal system, viz. the trigonometrical system, and so we shall study the general theory only in so far as it bears relation on this system [1]).

If an orthogonal system is to be at all useful for the development of functions, it should be *complete*, that is, whatever function ψ is added to $\{\varphi_m\}$, the new system ceases to be orthogonal. In fact, otherwise there would exist a function, just the function ψ, not vanishing identically, whose Fourier series with respect to $\{\varphi_n\}$ would consist entirely of zeros.

1.31. The notion of orthogonality, and hence that of Fourier coefficients and Fourier series, may be extended to the case of complex φ_n. We need only modify conditions 1.3(1) slightly, by replacing the products $\varphi_m \varphi_n$ by $\varphi_m \overline{\varphi_n}$, or, what is the same thing, by $\overline{\varphi_m} \varphi_n$ [2]). Similarly in (2) we replace $f\varphi_n$ by $f\overline{\varphi_n}$.

1.32. Rademacher's system. The following very instructive orthogonal and normal system was first considered by Rademacher [3]): $\varphi_n(x) = \text{sign}\sin(2^{n+1}\pi x)$ $(0 \leqslant x \leqslant 1)$. The function $\varphi_n(x)$ assumes alternately the values ± 1 in the interior of the intervals $(0, 2^{-n-1})$, $(2^{-n-1}, 2.2^{-n-1})$, ... The proof of orthogonality is very simple and may be left to the reader. The system is not complete, since e. g. the function $\psi(x) \equiv 1$ may be added to it.

---

[1]) We refer the reader interested in wider problems to a book by Kaczmarz and Steinhaus which is to appear in this series.

[2]) We denote by $\overline{z} = x - iy$ the number conjugate to $z = x + iy$. However the bar will also be used to denote the conjugate series, functions etc, where the word 'conjugate' has a different meaning. No misunderstanding will occur if the reader takes into account the context.

[3]) Rademacher [1]. See also Kaczmarz and Steinhaus [1].

1.4. The trigonometrical system. The system of functions 1, $\cos x$, $\sin x$, $\cos 2x$, $\sin 2x$, ..., i. e. the trigonometrical system, is orthogonal in $(-\pi, \pi)$. In fact, let $I_{m,n} = \int_{-\pi}^{\pi} \sin mx \sin nx \, dx$, and let $I'_{m,n}$, $I''_{m,n}$ denote the corresponding integrals with $\cos mx \sin nx$ and $\cos mx \cos nx$. Integrating the formula $2 \sin mx \sin nx = \cos(m-n)x - \cos(m+n)x$ and taking into account the periodicity of trigonometrical functions, we find that $I_{m,n} = 0$ whenever $m \neq n$. Similarly $I'_{m,n} = 0$, $I''_{m,n} = 0$, the former result being true even when $m = n$. The λ's are now $2\pi, \pi, \pi, \ldots$, and so, if for a given f we put

$$(1) \qquad a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos kt \, dt, \quad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin kt \, dt,$$

the Fourier series of f may be written in the form 1.1(1). Changing the definition of the preceding paragraph slightly in the case of a_0, we shall call a_k, b_k the Fourier coefficients of f. We shall denote by $\mathfrak{S}[f]$ the Fourier series of f and by $\bar{\mathfrak{S}}[f]$ the conjugate series. It is obvious that, if μ_1, μ_2 are two constants, then $\mathfrak{S}[\mu_1 f_1 + \mu_2 f_2] = \mu_1 \mathfrak{S}[f_1] + \mu_2 \mathfrak{S}[f_2]$.

1.41. *If a series* 1.1(1) *converges uniformly to a function* $f(x)$, *the coefficients* a_k, b_k *are given by the formulae* 1.4(1). The proof is the same as that which led to the formula 1.3(2).

1.42. If the function f is even, that is if $f(-x) = f(x)$, the coefficients b_k vanish and the integral defining a_k may be replaced by twice the integral over the interval $(0, \pi)$. If f is odd, that is if $f(-x) = -f(x)$, then $a_k = 0$ and the second integral in (1) may be replaced by twice the integral over $(0, \pi)$.

1.43. The complex form of Fourier series. The system of complex functions e^{kix} $(k = 0, \pm 1, \pm 2, \ldots)$ is orthogonal in $(-\pi, \pi)$. Putting

$$(1) \qquad c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} \, dt \qquad (k = 0, \pm 1, \ldots),$$

we may write the Fourier series, with respect to this system, in the form 1.13(1). Let us suppose, as we always shall do, except when it is stated otherwise, that f is real, and put $2c_k = a_k - ib_k$. Then a_k, b_k are given by 1.4(1) and we see that this Fourier series is equivalent to the trigonometrical Fourier series. However the complex form is very convenient and we shall frequenly use it.

1.44. It is also convenient to suppose that the functions whose Fourier series we consider are defined not only in $(-\pi, \pi)$, but for all real x by the condition of periodicity: $f(x+2\pi) = f(x)$, and, unless a statement to the contrary is made, we shall always assume this. Hence, we assume, in particular, that $f(-\pi) = f(\pi)$, a condition which we may always suppose satisfied [1]). Whenever we say that a series is the Fourier series of a continuous function f, we mean that f is continuous in $(-\infty, +\infty)$.

It is obvious that if a function $\psi(x)$ is of period 2π, the integrals of ψ, taken over arbitrary intervals of length 2π, are all equal. In particular, in 1.4(1) we may integrate over the interval $(0, 2\pi)$.

1.45. However, sometimes it is more convenient to consider the trigonometrical system not in $(0,2\pi)$ but in another interval, e. g. in $(0,1)$. The system $\{e^{2\pi ikx}\}$ is orthogonal and normal in the latter interval, so that the complex Fourier coefficients assume now the form

$$c_k = \int_0^1 f(t)\, e^{-2\pi ikt}\, dt \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

1.46. Integration and Fourier series. The problems of the theory of Fourier series are closely connected with the notion of integration. In the preceding definitions we assumed tacitly that the products $f \cos kx$, $f \sin kx$ were integrable. Hence we may consider Fourier-Riemann, Fourier-Lebesgue, Fourier-Denjoy series, according to the way in which the integrals are defined [2]). Except when otherwise stated, integrals are always Lebesgue integrals. It is assumed that the reader knows the elements of the Lebesgue theory of integration. Proofs of results of a more special character will be given in the text [3]).

Every integrable function $f(x)$ $(0 \leqslant x \leqslant 2\pi)$ has its Fourier series. It is even sufficient for f to be defined almost everywhere in $(0, 2\pi)$, i. e. everywhere, except in a set of measure 0. Two

---

[1]) See § 1.46.

[2]) For a general discussion see Lusin [1], [2].

[3]) The few passages in which the Denjoy integral is mentionned are not essential and may be omitted.

functions f_1 and f_2 which are equal almost everywhere have the same Fourier series and, following the usage of the Lebesgue theory, we call them equivalent: $f_1(x) \equiv f_2(x)$ and do not distinguish them from each other.

1.47. Fourier-Stieltjes series. Let $F(x)$ be a function of bounded variation, defined in $(0, 2\pi)$. Consider the series 1.1(1) with coefficients given by the formulae

$$(1) \qquad a_k = \frac{1}{\pi}\int_0^{2\pi} \cos kt\, dF(t), \quad b_k = \frac{1}{\pi}\int_0^{2\pi} \sin kt\, dF(t),$$

the integrals being Riemann-Stieltjes integrals. We shall write

$$(2) \qquad dF(x) \sim \tfrac{1}{2} a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

and call the series on the right the *Fourier-Stieltjes series* of dF. If F is absolutely continuous and $F'(x) = f(x)$, then $\mathfrak{S}[dF] = \mathfrak{S}[f]$.

It is convenient to define $F(x)$ for all x by the condition $F(x+2\pi) - F(x) = F(2\pi) - F(0)$. We may then integrate in the formulae (1) over any interval of length 2π. A necessary and sufficient condition for F to be periodic is: $\pi a_0 = F(2\pi) - F(0) = 0$. It follows that the function $F(x) - a_0 x/2$ is periodic.

1.5. *The trigonometrical system is complete.* This result is a simple corollary of theorems which we encounter later, but the following elementary proof, due to Lebesgue, is interesting in itself. Suppose first that there is a continuous and periodic $f \not\equiv 0$, whose Fourier coefficients all vanish. It follows that

$$(1) \qquad \int_{-\pi}^{\pi} f(x) T_n(x)\, dx = 0$$

for every trigonometrical polynomial T_n [1]). We may suppose without loss of generality that there exists a point x_0 and two numbers $\varepsilon, \delta > 0$, such that $f(x) > \varepsilon$ for $x \in I = (x_0 - \delta, x_0 + \delta)$ [2]). It will be enough to show that there exists a sequence $\{T_n(x)\}$, such

---

[1]) Trigonometrical polynomials of order n are finite sums of the form $\frac{1}{2}\alpha_0 + (\alpha_1 \cos x + \beta_1 \sin x) + \ldots + (\alpha_n \cos nx + \beta_n \sin nx)$.

[2]) $x \in A$ means: x belongs to a set A; $x \bar{\in} A$ means: x does not belong to A; $A \subset B$ means: A is a subset of B.

that (i) $T_n(x) \geqslant 1$ in I, (ii) $T_n(x)$ tends uniformly to $+\infty$ in every interval I' interior to I, (iii) $T_n(x)$ are uniformly bounded outside I (mod 2π). For the left-hand side of (1) is the sum of two integrals, extended respectively over I and the rest of $(-\pi, \pi)$. The first of them exceeds $|I'|$. Max $T_n(x)$ $(x \in I') \to \infty$ [1]. The second is bounded and so (1) is impossible for large n. We put $T_n = t^n$, where $t(x) = 1 + \cos(x - x_0) - \cos\delta$. In this case $t(x) \geqslant 1$ in I, $t(x) > 1$ in I', $|t(x)| \leqslant 1$ for $x \,\bar{\epsilon}\, I$ (mod 2π).

Suppose now f only integrable and let $F(x)$ be the integral of f over $(-\pi, x)$. Hence $F(-\pi) = 0$, and the condition $a_0 = 0$ involves $F(\pi) = 0$. Integrating 1.4(1) by parts we obtain $A_1 = B_1 = A_2 = B_2 = \ldots = 0$, where $A_0, A_1, B_1, \ldots$ are the Fourier coefficients of F. Hence, for a suitable constant c, the continuous function $F - c$ will have all its Fourier coefficients equal to 0, and so $F(x) = c$. Since $F(-\pi) = 0$, we obtain ultimately $F(x) = 0$, i. e. $f = 0$. The reader will observe that the proof remains valid with more general definitions of an integral than that of Lebesgue.

1.51. Corollaries. (i) *If f_1 and f_2 have the same Fourier series then $f_1 = f_2$.* (ii) *If, for f continuous, $\mathfrak{S}[f]$ converges uniformly, it converges to f.* Let $g(x)$ denote the sum of $\mathfrak{S}[f]$. Then the coefficients of $\mathfrak{S}[f]$ are the Fourier coefficients of g (see § 1.41), and so $f = g$.

1.6. Bessel's inequality. Parseval's relation. We may also be led to the notion of Fourier coefficients by the following considerations. Let $\{\varphi_n\}$ be a system of functions orthogonal and normal in an interval (a, b), and let f be a function such that f^2 is integrable in (a, b). We fix an integer $n \geqslant 0$, put $T = \gamma_0\varphi_0 + \gamma_1\varphi_1 + \ldots + \gamma_n\varphi_n$ and then ask what values of the constants $\gamma_0, \gamma_1, \ldots \gamma_n$ make the integral

$$(1) \quad \int_a^b (f - T)^2\,dx = \int_a^b (f^2 - 2fT + T^2)\,dx = \int_a^b f^2\,dx - 2\sum_{k=0}^{n} c_k\gamma_k + \sum_{k=0}^{n}\gamma_k^2$$

a minimum, $c_0, c_1, \ldots$ being the Fourier coefficients of f. The last two sums can be written as $-\gamma_0(2c_0 - \gamma_0) - \ldots - \gamma_n(2c_n - \gamma_n)$ and since the function $u(\alpha - u)$ assumes its maximum when $u = \alpha/2$, we see that *the left-hand side of* (1), *which is called the quadratic*

[1]) $|E|$ denotes the measure of a set E.

approximation to f by T, is a minimum when $\gamma_k = c_k$ ($k = 0, 1, \dots n$), that is when T is the n-th partial sum of the Fourier series of f [1].

Putting $\gamma_k = c_k$ and taking into account that the integral on the left in (1) is non-negative, we obtain the very important relation

$$\text{(2)} \qquad \sum_{k=0}^{n} c_k^2 \leqslant \int_a^b f^2\, dx,$$

which is called 'Bessel's inequality'. Since n in (2) is arbitrary, we have also:

$$\text{(3)} \qquad \sum_{k=0}^{\infty} c_k^2 \leqslant \int_a^b f^2\, dx.$$

For some systems $\{\varphi_n\}$ the sign $\leqslant$ in (3) may be replaced by $=$ and the equation we then obtain is called 'Parseval's relation'.

Since the system $1/\sqrt{2\pi}$, $(\cos x)/\sqrt{\pi}$, $(\sin x)/\sqrt{\pi}, \dots$ is orthogonal and normal, we obtain from (3), using the notation 1.4(1), that

$$\text{(4)} \qquad \tfrac{1}{2} a_0^2 + \sum_{k=1}^{\infty} (a_k^2 + b_k^2) \leqslant \frac{1}{\pi} \int_0^{2\pi} f^2 dx,$$

for any f with integrable square.

Corollary. If f^2 is integrable, then $a_k \to 0$, $b_k \to 0$.

1.61. The argument used in § 1.6 shows that, if $\mathfrak{S}[f]$ converges uniformly, in particular, if f is a trigonometrical polynomial, there is equality in (4).

1.7. Remarks on series and integrals. It will be convenient to collect here a few elementary theorems on series and integrals, which will often be used in the sequel. Let $f(x)$ and $g(x) > 0$ be two functions defined for $x > x_0$. We say that $f(x) = o(g(x))$ if $f(x)/g(x) \to 0$ as $x \to \infty$. If $f(x)/g(x)$ is bounded for all x sufficiently large, we write $f(x) = O(g(x))$. The same notation is used when x tends to a finite limit, or to $-\infty$, or even when x tends to its limit through a discrete sequence of values. In particular, an expresion is $o(1)$ or $O(1)$ if it tends to 0 or is bounded, as the case may be.

---

[1] Toepler [1].

Two functions $f(x)$ and $g(x)$ will be called asymptotically equal in the neighbourhood of x_0 if $f(x)/g(x) \to 1$ as $x \to x_0$, and we write $f(x) \simeq g(x)$. If there exist two constants $A > 0$, $B > 0$, such that $A \leqslant f(x)/g(x) \leqslant B$ for x sufficiently near x_0, we shall say that f and g are of the same order in the neighbourhood of x_0 and write $f(x) \sim g(x)$. Similar definitions and notations will be used for sequences.

Examples: $x = O(x^2)$ as $x \to \infty$, $x^2 = o(x)$ as $x \to 0$, $\log r = O(|1-r|)$ as $r \to 1$, $n^{-1} = o(1)$ as $n \to \infty$, $n + \sqrt{n} \simeq n$ as $n \to \infty$, $\exp n \sim \exp(n + \sin n)$ as $n \to \infty$ [1]).

1.71. *Let $f(x)$ and $g(x) > 0$ be two functions defined for $a \leqslant x < b$ and integrable in any interval $(a, b-\varepsilon)$. Let $F(x)$ and $G(x)$ be the integrals of f, g over (a, x). If $f(x) = o(g(x))$ and $G(x) \to \infty$ as $x \to b$, then $F(x) = o(G(x))$.* Suppose that $|f(x)|/g(x) < \varepsilon/2$ for $a < x_0 \leqslant x < b$. For such values of x we have the inequality

$$|F(x)| \leqslant \int_a^{x_0} |f|\,dt + \int_{x_0}^{x} |f|\,dt \leqslant \int_a^{x_0} |f|\,dt + \frac{\varepsilon}{2} G(x).$$ Since $G(x) \to \infty$, the last sum is less than $\varepsilon G(x)$ for $x \geqslant x_1$ $(x_0 \leqslant x_1 < b)$ and, since ε is arbitrary, the theorem follows.

1.72. In the above theorem the rôle played by a and b can, obviously, be reversed. If $a = 0$, $b = \infty$, it has an analogue for finite sums: *Let f_n and $g_n > 0$ be two sequences, $F_n = f_0 + \ldots + f_n$, $G_n = g_0 + \ldots + g_n$. If $f_n = o(g_n)$, $G_n \to \infty$, then $F_n = o(G_n)$.* The proof is essentially the same as for integrals.

1.73. The proof of the following result is still simpler. *If the series $f_0 + f_1 + \ldots$, $g_0 + g_1 + \ldots$, $g_n > 0$,* converge and if, $F_n = f_n + f_{n+1} + \ldots$, $G_n = g_n + g_{n+1} + \ldots$, then $f_n = o(g_n)$ implies $F_n = o(G_n)$.

1.74. *Let $f(x)$ $(x \geqslant 0)$ be a positive, finite, monotonic function. Let $F(x)$ be the integral of f over $(0, x)$ and $F_n = f(0) + f(1) + \ldots + f(n)$. Then* (i) *if f is decreasing, $F(n) - F_n$ tends to a finite limit C,* (ii) *if f increases, then $F(n) \leqslant F_n \leqslant F(n) + f(n)$.* In order to prove (i) we observe that, from geometrical considerations, we may write $f(k) \leqslant F(k) - F(k-1) \leqslant f(k-1)$ or, what is the same thing, $0 \leqslant F(k) - F(k-1) - f(k) \leqslant f(k-1) - f(k)$, $k = 1, 2, \ldots$. Since

---

[1]) $\exp x$ means e^x.

the series with terms $f(k-1)-f(k)$ converges, the same may be said of the series with terms $F(k)-F(k-1)-f(k)$ and partial sums $F(n)-F_n+f(0)$.

For example, the difference $1+1/2+\dots+1/n-\log n$ tends to a constant C, usually called Euler's constant.

To obtain (ii) we proceed similarly, summing the inequalities $f(k-1)\leqslant F(k)-F(k-1)\leqslant f(k)$ from $k=1$ to n.

1.741. *If either $f(x)$ decreases and $F(x)\to\infty$, or $f(x)$ increases and $f(x)/F(x)\to 0$, then $F_n\simeq F(n)$.*

1.742. If $f(x)>0$ is decreasing and integrable over $(0,\infty)$, $F(x)$ denotes the integral of f over (x,∞), and $F_n=f(n)+f(n+1)+\dots$, then $0\leqslant F_n-F(n)\leqslant f(n)$. In particular, if $f(x)/F(x)\to 0$, we have $F_n\simeq F(n)$.

In cases when $F(n)$ can be easily obtained, the above theorems give us approximate expressions for F_n.

Examples: $\sum_{k=1}^{n} k^{\alpha}\simeq\frac{n^{\alpha+1}}{\alpha+1}$, $\sum_{k=n}^{\infty} k^{-\beta}\simeq\frac{n^{-\beta+1}}{\beta-1}$ $(\alpha>-1,\ \beta>1)$.

1.8. Miscellaneous theorems and examples.

1. Show that $\sin x+{}^1/_2\sin 2x+{}^1/_3\sin 3x+\dots$ converges to $(\pi-x)/2$ in the interior, and to 0 at the ends, of $(0,2\pi)$.

2. Let (i) $f_1(x)$, $|x|\leqslant\pi$, be even, equal to 1 in $(0,h)$ and to 0 in (h,π), $0<h<\pi$, (ii) $f_2(x)$, $|x|\leqslant\pi$, be even, continuous, vanishing in $(2h,\pi)$, $0<h\leqslant\pi/2$, equal to 1 at $x=0$, and linear in $(0,2h)$, (iii) $f_3(x)=\operatorname{sign}x$, $|x|<\pi$, (iv) $\varphi(x)=(\pi-x)/2$, $0<x<2\pi$, (v) $F(x)=\pi[x/2\pi]$ [1]). Show that

$$f_1(x)\sim\frac{2h}{\pi}\left[\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{\sin nh}{nh}\right)\cos nx\right],\qquad f_2(x)\sim\frac{2h}{\pi}\left[\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{\sin nh}{nh}\right)^2\cos nx\right],$$

$$f_3(x)\sim\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)x}{2n+1},\qquad \varphi(x)\sim\sum_{n=1}^{\infty}\frac{\sin nx}{n}$$

$$dF(x)\sim\frac{1}{2}+\sum_{n=1}^{\infty}\cos nx,\qquad |\sin x|=\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{\sin^2 nx}{4n^2-1}.$$

3. Let $f(x)\not\equiv 0$ be even, $g(x)\not\equiv 0$ odd, both non-negative in $(0,\pi)$, and let $a_0,a_1,\dots,b_1,b_2,\dots$ be the Fourier coefficients of f and g respectively. Show that $|a_m|<a_0$, $|b_n|<nb_1$, $m=1,2,\dots$, $n=2,3,\dots$

[Prove, by induction, the inequality $|\sin nt|\leqslant n|\sin t|$. Carathéodory [1], Rogosinski [1]].

---

[1]) $[y]$ denotes the integral part of y.

4. Each of the systems 1, $\cos x$, $\cos 2x$, ... and $\sin x$, $\sin 2x$, ... is orthogonal and complete in $(0, \pi)$.

5. Let $\{\varphi_n\}$ denote Rademacher's system. Put $\chi_0(t) = 1$, $\chi_N(t) = \varphi_{n_1}(t)\,\varphi_{n_2}(t) \ldots \varphi_{n_k}(t)$, if $N = 2^{n_1} + 2^{n_2} + \ldots + 2^{n_k}$. Show that the system $\{\chi_N\}$ is orthogonal normal and complete in $(0, 1)$.

[If $\int_0^1 f(t) \prod_{k=0}^{n} (1 + \varphi_k(x)\,\varphi_k(t))\, dt = 0$ for every x and n, and if F is an integral of f then $F'(x) = 0$ at almost every x. The system $\{\chi_N\}$ was first considered by Walsh [1]; see also Kaczmarz [1], Paley [1]].

6. Orthogonal and normal systems may be defined also in spaces of higher dimensions, the interval of integration being replaced by any measurable set. Show that if $\{\varphi_m(x)\}$ and $\{\psi_n(y)\}$ are orthogonal, normal and complete in the intervals $a \leqslant x \leqslant b$, $c \leqslant y \leqslant d$ respectively, then the doubly infinite system $\{\varphi_m(x)\,\psi_n(y)\}$ is orthogonal, normal and complete in the rectangle R with opposite corners at the points (a, c), (b, d).

[If $\iint_R f(x,y)\,\varphi_m(x)\,\psi_n(y)\,dx\,dy = 0$ for all m, n, the functions $f_m(y) = \int_a^b f(x, y)\,\varphi_m(x)\,dx$ vanish for almost every y, and so $f(x, y)$ vanishes almost everywhere on almost every line $y = \text{const}$].

CHAPTER II.

Fourier coefficients. Tests for the convergence of Fourier series.

2.1. Operations on Fourier series. We begin by proving a few theorems which show that certain formal operations on Fourier series are legitimate.

If $f(x) \sim \sum_{-\infty}^{+\infty} c_m e^{imx}$ *and* u *is constant, then we have*

$$f(x+u) \sim \sum_{-\infty}^{+\infty} c_m e^{imu} e^{imx} = \tfrac{1}{2} a_0(u) + \sum_{m=1}^{\infty} (a_m(u) \cos mx + b_m(u) \sin mx),$$

where $a_m(u) = a_m \cos mu + b_m \sin mu$, $b_m(u) = b_m \cos mu - a_m \sin mu$.

In fact, $\dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} f(x+u) e^{-imx}\, dx = \dfrac{e^{imu}}{2\pi} \displaystyle\int_0^{2\pi} f(x+u) e^{-im(x+u)}\, dx = e^{imu} c_m$.

2.11. *Let* $f(x) \sim \sum_{-\infty}^{+\infty} c_m e^{imx}$, $g(x) \sim \sum_{-\infty}^{+\infty} d_m e^{imx}$. *Then*

$$(1) \qquad h(x) = \frac{1}{2\pi} \int_0^{2\pi} f(x+t)\, g(t)\, dt \sim \sum_{-\infty}^{+\infty} c_m d_{-m} e^{imx}.$$

More precisely: under the conditions of the theorem, (i) the function $h(x)$ exists for almost all x and is integrable, (ii) its Fourier coefficients are $c_m d_{-m}$ [1]). The formulae in (1) are obtained by term-by-term integration of the product of the Laurent series for $\mathfrak{S}[f(x+t)]$ and $\mathfrak{S}[g]$.

To prove (i) it is sufficient to suppose that $f \geqslant 0$, $g \geqslant 0$. Let $f_n(x) = \mathrm{Min}\,(f(x), n)$, $g_n(x) = \mathrm{Min}\,(g(x), n)$, and let $h_n(x)$ be the function obtained from f_n, g_n by means of (1). Using Fubini's

---

[1]) W. H. Young [1].

well known theorem on the inversion of the order of integration, we have

$$(2)\qquad \begin{aligned}\int_0^{2\pi} h_n\,dx &= \int_0^{2\pi} dx \int_0^{2\pi} f_n(x+t)\,g_n(t)\,dt = \\ \int_0^{2\pi} g_n(t)\left[\int_0^{2\pi} f_n(x+t)\,dx\right]dt &= \int_0^{2\pi} f_n(x)\,dx \int_0^{2\pi} g_n(x)\,dx.\end{aligned}$$

Since $\{f_n(t)\,g_n(x+t)\}$ is increasing and tends to $f(t)\,g(x+t)$, it follows that $\{h_n(x)\}$ is also increasing and tends to $h(x)$. Hence, making $n\to\infty$, we find from (2) that $h(x)$ is integrable, and, in particular, finite almost everywhere.

Using Fubini's theorem again, we have

$$\frac{1}{2\pi}\int_0^{2\pi} h(x)\,e^{-imx}\,dx = \frac{1}{4\pi^2}\int_0^{2\pi} g(t)\,e^{imt}\left[\int_0^{2\pi} f(x+t)\,e^{-im(x+t)}\,dx\right]dt = c_m\,d_{-m}.$$

We leave it to the reader to rearrange $\mathfrak{S}[h]$ in the form with real coefficients.

2.12. Differentiation of Fourier series. Suppose that $f(x)$ is an integral, i. e. is absolutely continuous. Integrating by parts, we have, for $m\neq 0$,

$$(1)\qquad c_m = \frac{1}{2\pi}\int_0^{2\pi} f\,e^{-imx}\,dx = \frac{1}{2\pi i m}\int_0^{2\pi} f'\,e^{-imx}\,dx = \frac{c'_m}{im},$$

or $c'_m = imc_m$, c'_m being the Fourier coefficient of f'. Since f is periodic, we find that $c'_0 = 0$. In other words, if $\mathfrak{S}'[f]$ denotes the result of differentiating $\mathfrak{S}[f]$ term by term, we have $\mathfrak{S}'[f] = \mathfrak{S}[f']$:

$$f' \sim i\sum_{-\infty}^{+\infty} mc_m\,e^{imx} = \sum_{m=1}^{\infty} m\,(b_m\cos mx - a_m\sin mx).$$

If f is a k-th integral, then $\mathfrak{S}^{(k)}[f] = \mathfrak{S}[f^{(k)}]$.

2.13. Suppose that f has a number of simple discontinuities at points $0\leqslant x_1 < x_2 < \ldots < x_k < 2\pi$ and that it is absolutely continuous in the interior of each interval (x_i, x_{i+1}). *Let us put*

$d_i = [f(x_i+0) - f(x_i-0)]/\pi$. *Then* $\mathfrak{S}'[f] - \mathfrak{S}[f'] = d_1 D(x-x_1) + \dots + d_k D(x-x_k)$, *where* $D(x) = \frac{1}{2} + \cos x + \cos 2x + \dots$[1].

Let $\varphi(x)$ be periodic and equal to $(\pi - x)/2$ for $0 < x < 2\pi$, $\varphi(0) = \varphi(2\pi) = 0$. Since $d_i \varphi(x - x_i)$ has at x_i the same jump as $f(x)$, the difference $g(x) = f(x) - \Phi(x)$, where $\Phi(x) = d_1 \varphi(x - x_1) + \dots + d_k \varphi(x - x_k)$, is everywhere continuous, indeed absolutely continuous. Moreover, except at the points x_i, $g' - f' = (d_1 + \dots + d_k)/2 = C$. Now $\mathfrak{S}'[f] = \mathfrak{S}'[\Phi] + \mathfrak{S}'[g] = \mathfrak{S}'[\Phi] + \mathfrak{S}[g'] = \mathfrak{S}'[\Phi] + \mathfrak{S}[f' + C] = \mathfrak{S}[f'] + \mathfrak{S}'[\Phi] + C$. Taking into account the particular form of C and $\mathfrak{S}'[\Phi]$ (§ 1.8, 2 (iv)), the result follows.

2.14. Let $F(x)$ be a function of bounded variation, so that, if c_m are the complex coefficients of $\mathfrak{S}[dF]$, the difference $F - c_0 x$ is periodic (§ 1.45). Let C_m be the Fourier coefficients of the latter function. Then, for $m \neq 0$,

$$C_m = \frac{1}{2\pi}\int_0^{2\pi} (F - c_0 x)\, e^{-imx}\, dx = \frac{1}{2\pi i m}\int_0^{2\pi} e^{-imx}\, d(F - c_0 x) = \frac{c_m}{im}.$$

Let us agree to write

$$F(x) \sim c_0 x + C_0 + \sum_{-\infty}^{+\infty}{}' \frac{c_m}{im} e^{imx}, \text{ instead of } F(x) - c_0 x \sim C_0 + \sum_{-\infty}^{+\infty}{}' \frac{c_m}{im} e^{imx},$$

where $'$ denotes that the term for which $m = 0$ is omitted, i. e. we represent F as the sum of a linear and a periodic function. Then $\mathfrak{S}[dF]$ is obtained by formal differentiation of the former series, that is, *the class of Fourier-Stieltjes series, and that of formally differentiated Fourier series of functions of bounded variation are identical.*

2.15. Integration of Fourier series. Let f be periodic and F an integral of f. Since $F(x + 2\pi) - F(x)$ is equal to the integral of f over $(x, x + 2\pi)$, or, what is the same thing, over $(0, 2\pi)$, a necessary and sufficient condition for the periodicity of F is that the constant term of $\mathfrak{S}[f]$ should vanish. Suppose

---

[1]) The series $D(x)$, which is very important in the theory of Fourier series, diverges everywhere. However, it is summable to 0, for example by Abel's method, if $x \neq 0 \pmod{2\pi}$.

this condition satisfied. Then $\mathfrak{S}[f]$ is obtained by formal differentiation of $\mathfrak{S}[F]$, i. e.

$$(1)\quad F(x) \sim C + \sum_{-\infty}^{+\infty}{}' \frac{c_m}{im} e^{imx} = C + \sum_{m=1}^{\infty} \frac{a_m \sin mx - b_m \cos mx}{m}.$$

Here C is the constant of integration and depends on the choice of F. If $c_0 \neq 0$, the periodic function $F - c_0 x$ is an integral of $f - c_0$ and the series in (1) is $\mathfrak{S}[F - c_0 x]$.

Example. Let $f_0(x), f_1(x), \dots f_k(x), \dots$, $0 < x < 2\pi$, be the functions defined by the conditions (i) $f_0(x) = -1$, (ii) $f'_k(x) = f_{k-1}(x)$, (iii) the integral of f_k over $(0, 2\pi)$ vanishes, $k = 1, 2, \dots$ The reader will easily verify that $f_k(x) \sim \sum_{m=-\infty}^{+\infty}{}' \frac{e^{imx}}{i^k \; m^k}$. In the interval $(0, 2\pi)$ the function $f_k(x)$ a is polynomial of order k.

2.2. Modulus of continuity. Let $f(x)$ be a function defined for $a \leqslant x \leqslant b$; let $\omega(\delta) = \omega(\delta; f) = \text{Max}\,|f(x_1) - f(x_2)|$ for all x_1, x_2 belonging to (a, b) and such that $|x_1 - x_2| \leqslant \delta$. The function $\omega(\delta)$ is called the *modulus of continuity* of f [1]) and this notion is very useful in the theory of Fourier series. The function f is continuous if and only if $\omega(\delta) \to 0$ with δ. If $\omega(\delta) < C\delta^\alpha$, where $0 < \alpha \leqslant 1$ and C denotes a number independent of δ, we say that f satisfies the *Lipschitz condition* of order α, or $f \in \text{Lip}\,\alpha$, in (a, b). The restriction $\alpha \leqslant 1$ is quite natural, since if $\omega(\delta)/\delta \to 0$ with δ, $f'(x)$ exists and is equal to 0 everywhere, so that $f = \text{const}$.

Suppose now for simplicity that (a, b) coincides with $(0, 2\pi)$ and consider a periodic and integrable function f, not necessarily continuous. Let $\omega_1(\delta) = \omega_1(\delta; f) = \text{Max} \int_0^{2\pi} |f(x+h) - f(x)|\, dx$ for all $0 < h \leqslant \delta$. The function $\omega_1(\delta)$ will be called the *integral modulus of continuity* of f.

2.201. *For every integrable* f, $\lim \omega_1(\delta; f) = 0$ *as* $\delta \to 0$. Given a function g, let $I(g)$ denote the integral of $|g|$ over $(0, 2\pi)$. If for any $\varepsilon > 0$ we have $f = f_1 + f_2$, where $\omega_1(\delta; f_1) \to 0$ with δ, and $I(f_2) < \varepsilon$, then $\omega_1(\delta; f) \to 0$. In fact: $\omega_1(\delta; f) \leqslant \omega_1(\delta; f_1) + \omega_1(\delta; f_2) \leqslant \omega_1(\delta; f_1) + 2\,I(f_2) < 3\varepsilon$, if $0 < \delta \leqslant \delta_0(\varepsilon)$. Now the theorem is

---

[1]) Lebesgue [1]

certainly true when E is the characteristic function of a set E [1]) consisting of a finite number of intervals, hence it is true also when E is an arbitrary open set, and consequently when E is measurable. It follows that the theorem holds when f assumes only a finite number of values, hence when f is bounded, and finally when f is integrable.

2.21. *If c_m are the complex Fourier coefficients of a function f, then*

$$(1) \qquad |c_m| \leqslant \tfrac{1}{2}\,\omega\left(\frac{\pi}{m}\right), \quad |c_m| \leqslant \frac{1}{4\pi}\,\omega_1\left(\frac{\pi}{m}\right).$$

Replacing x by $x+\pi/m$ in the integral defining c_m, we have that $2\pi c_m$ is equal to

$$\int_0^{2\pi} f(x)\,e^{-imx}\,dx = -\int_0^{2\pi} f\left(x+\frac{\pi}{m}\right)e^{-imx}\,dx = \tfrac{1}{2}\int_0^{2\pi}\left[f(x)-f\left(x+\frac{\pi}{m}\right)\right]e^{-imx}\,dx$$

and the last integral does not exceed either $\pi\omega(\pi/m)$ or $\frac{1}{2}\,\omega_1(\pi/m)$ in absolute value.

2.211. The Riemann-Lebesgue theorem. *The Fourier coefficients of integrable functions tend to* 0. This follows from Theorem 2.201 and the second formula 2.21(1). A slightly simpler proof runs as follows: $f=f_1+f_2$, where f_1 is bounded and $I(f_2)<\varepsilon$. ($I(f)$ has the same meaning as in § 2.201). Correspondingly, $c_m = c'_m + c''_m$, where $|c''_m| \leqslant I(f_2)/2\pi < \varepsilon/2\pi$ and $c'_m \to 0$ (§ 1.6, Corollary). Hence $|c_m| \leqslant |c'_m| + |c''_m| < \varepsilon$ for $m > m_0$.

2.212. *If $f \in \text{Lip}\,\alpha$ $(0<\alpha\leqslant 1)$, then $c_m = O(m^{-\alpha})$* [2]). Here O cannot be replaced by o (§ 2.9.3), except in the case $\alpha=1$. In this case, since f is absolutely continuous, the differentiated $\mathfrak{S}[f]$ is still a Fourier series, so that $c_m = o(m^{-1})$.

2.213. *If f is of bounded variation, then $|a_m| \leqslant V/m$, $|b_m| \leqslant V/m$, $m=1,2,\dots$, where V denotes the total variation of f over* $(0, 2\pi)$. Suppose first that f is non-decreasing and $\geqslant 0$. Using the second mean-value theorem we have

$$\pi a_m = \int_0^{2\pi} f(x)\cos mx\,dx = f(2\pi)\int_\xi^{2\pi}\cos mx\,dx, \quad 0<\xi<2\pi,$$

---

1) The function equal to 1 in a set E and to 0 elsewhere is called the characteristic function of E.

2) Lebesgue [1].

and so $|a_m| \leqslant 2f(2\pi)/\pi m$. In the general case, since $f = f_1 - f_2$, where f_1, f_2 are respectively the positive and negative variations of f, we find that $|a_m|$, and similarly $|b_m|$, does not exceed $2[f_1(2\pi) + f_2(2\pi)]/\pi m = 2V/\pi m \leqslant V/m$.

The result can also be stated in the following form: *the coefficients of* $\mathfrak{S}[df]$ *form a bounded sequence.* The simplest examples show that this result cannot be improved (§ 1.8.2(v)). The fact that it cannot be improved even when f is of bounded variation *and* continuous lies much deeper. We state without proof the following result, which will be established in Ch. XI. Let C be the well-known ternary set of Cantor constructed on $(0, 2\pi)$. *If* $f(x)$ *is any function constant in each of the intervals complementary to* C, *but not equivalent to a constant in* $(0, 2\pi)$, *the Fourier coefficients of* f *are not* $o(1/n)$.

Taking f continuous and of bounded variation we obtain the required example.

2.22. Fourier-Riemann coefficients. Theorem 2.211 is no longer true for Fourier-Riemann series. Let

$$f(x) = \frac{d}{dx}(x^{\nu} \cos 1/x) \quad 0 < \nu < \tfrac{1}{2}, \quad S(x) = \sum_{n=1}^{\infty} e^{in^{\alpha}} n^{\beta} e^{inx}.$$

It was shown by Riemann[1]) that the Fourier coefficients of the function f, which is integrable R, are not necessarily $o(1)$, and not even $o(n^{(1-2\nu)/4})$. It can also be proved that the real and imaginary parts of the series $S(x)$ are both Fourier-Riemann series, if only $0 < \alpha < 1$, $\beta < \alpha/2$ [2]). We will give here a stronger example, based on the fact that the integral of $\sin^2 nx$ over (a, b) tends to $(b-a)/2$ as $n \to \infty$.

2.221. *Given an arbitrary sequence of numbers* $\lambda_n \to \infty$, $\lambda_n = o(n)$, *there exists a function* f *integrable* R, *whose sine coefficients* b_n *exceed* λ_n *for infinitely many* n [3]).

Let $\lambda_n = \varepsilon_n n$, $\varepsilon_n \to 0$. We shall define a sequence of non-overlapping intervals $I_k = (\alpha_k/2, \alpha_k)$, $k = 1, 2, \ldots$, approaching the point 0 from the right. Let $f(x) = c_k \sin n_k x$ in I_k, and $f(x) = 0$ elsewhere in $(-\pi, \pi)$. The positive coefficients c_k and the integers $n_1 < n_2 < \ldots$ satisfy a series of relations; in particular (1) $n_k \alpha_k$ are multiples of 4π, so that f is continuous for $x \neq 0$ and the integral of f over I_k vanishes; (2) $c_k/n_k = 1/k \to 0$, which implies that f is integrable R over $(0, \pi)$. Let $n_1 = 4$, $c_1 = 4$, $I_1 = (\pi/2, \pi)$ and suppose we have defined n_i, c_i, I_i for $i = 1, 2, \ldots, k-1$ and consequently $f(x)$ for $\alpha_{k-1}/2 \leqslant x \leqslant \pi$. Put

---

1) Riemann [1].
2) This is implicitly contained in Hardy [1].
3) Titchmarsh [1].

$\alpha_k = 4\pi/p$, p being the smallest integer such that (3) $\alpha_k \leqslant 1/n_{k-1}$. A little attention shows that (3_1) $\alpha_k \geqslant \frac{1}{2} n_{k-1}$. Let n_k divisible by p be so large that (4) the integral of $\sin^2 n_k x$ over I_k exceeds $\alpha_k/8$, (5) the integral of $f \sin n_k x$ over (α_k, π) is less than 1 in absolute value, and, finally, (6) $4\varepsilon_{n_k} < 1/16\, k n_{k-1}$. To investigate the behaviour of the integral, extended over $(0, \pi)$, of the product $f(x) \sin n_k x$, we break up this integral into three, extended over $(0, \alpha_{k+1})$, $(\alpha_k/2, \alpha_k)$, (α_k, π), and denote them by A_k, B_k, C_k. We have $|C_k| \leqslant 1$ (cond. (5)), and, since $\sin n_k x$ is monotonic in $(0, \alpha_{k+1})$ (cond. (3)), the second mean-value theorem shows that $A_k \to 0$. In virtue of conditions (4), (2), (3_1), (6) we have $B_k > c_k \alpha_k/8 = n_k \alpha_k/8\,k \geqslant n_k\, 1/16\, k n_{k-1} > 4\varepsilon_{n_k} n_k = 4\lambda_{n_k}$. Therefore we have $\pi b_{n_k} = A_k + B_k + C_k > 4\lambda_{n_k} - 1 - o(1)$, i. e. $b_{n_k} > \lambda_{n_k}$ for k large, and the result follows.

2.222. Since integration by parts subsists for Denjoy's integrals, both special and general [1]), the argument of § 2.11 proves that Fourier-Denjoy series, which are obtained by term-by-term differentiation of $\mathfrak{S}[F]$, with F continuous, have coefficients $o(n)$. This result cannot be improved, as Theorem 2.221 shows.

2.3. Formulae for partial sums. The object of the rest of this chapter is to establish some conditions for the convergence of Fourier series and of the conjugate series. It will be convenient to treat these two problems side by side. If

$$(1)\quad \tfrac{1}{2} a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx), \quad \sum_{k=1}^{\infty} (a_k \sin kx - b_k \cos kx)$$

are $\mathfrak{S}[f]$ and $\bar{\mathfrak{S}}[f]$ respectively, the n-th partial sums, $s_n(x) = s_n(x; f)$ and $\bar{s}_n(x) = \bar{s}_n(x; f)$, of these series can be written in the following forms

$$s_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, dt +$$

$$(2)\quad + \frac{1}{\pi} \sum_{k=1}^{n} \left(\cos kx \int_{-\pi}^{\pi} f(t) \cos kt\, dt + \sin kx \int_{-\pi}^{\pi} f(t) \sin kt\, dt\right) =$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left(\tfrac{1}{2} + \sum_{k=1}^{n} \cos k(t-x)\right) dt = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\, D_n(t-x)\, dt =$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t+x)\, D_n(t)\, dt,$$

---

[1]) For the theory of these integrals we refer the reader to Saks's *Théorie de l'intégrale*, Ch. X.

$$\bar{s}_n(x) = -\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left(\sum_{k=1}^{n} \sin k(t-x)\right) dt = -\frac{1}{\pi}\int_{-\pi}^{\pi} f(t+x)\,\bar{D}_n(t)\,dt,$$

where $D_n(u) = \dfrac{\sin(n+\frac{1}{2})u}{2\sin\frac{1}{2}u}$, $\bar{D}_n(u) = \dfrac{\cos\frac{1}{2}u - \cos(n+\frac{1}{2})u}{2\sin\frac{1}{2}u}$ (§ 1.12).

The functions D_n and $\bar{D}_n$ are called 'Dirichlet's kernel', and 'Dirichlet's conjugate kernel' respectively. However, instead of considering s_n and $\bar{s}_n$, it will be slightly more convenient to consider the expressions $s_n^*(x) = s_n(x) - (a_n \cos nx + b_n \sin nx)/2$, $\bar{s}_n^*(x) = \bar{s}_n(x) - (a_n \sin nx - b_n \cos nx)/2$. Since the differences $s_n - s_n^*$ and $\bar{s}_n - \bar{s}_n^*$ tend uniformly to 0, this is completely justified. Putting

$$D_n^*(u) = D_n(u) - \tfrac{1}{2}\cos nu = \frac{\sin nu}{2\,\mathrm{tg}\,\frac{1}{2}u},$$

$$\bar{D}_n^*(u) = \bar{D}_n(u) - \tfrac{1}{2}\sin nu = \frac{1-\cos nu}{2\,\mathrm{tg}\,\frac{1}{2}u},$$

and arguing as before, we have

$$(3)\quad s_n^*(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\,D_n^*(t)\,dt,\quad \bar{s}_n^*(x) = -\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\,\bar{D}_n^*(t)\,dt.$$

If $f \equiv 1$, then $s_n^*(x) = 1$ for $n > 0$. Since $D_n^*(t)$ is even, $\bar{D}_n^*(t)$ odd, we have

$$s_n^*(x) - f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t+x)\,D_n^*(t)\,dt - \frac{f(x)}{\pi}\int_{-\pi}^{\pi} D_n^*(t)\,dt =$$

$$(4)\qquad = \frac{1}{\pi}\int_0^{\pi} \frac{\varphi(t)}{2\,\mathrm{tg}\,\frac{1}{2}t}\sin nt\,dt,$$

$$\bar{s}_n^*(x) = -\frac{1}{\pi}\int_0^{\pi} \frac{\psi(t)}{2\,\mathrm{tg}\,\frac{1}{2}t}(1-\cos nt)\,dt,$$

where $\varphi(t) = \varphi_x(t) = \varphi_x(t; f) = f(x+t) + f(x-t) - 2f(x)$, $\psi(t) = \psi_x(t) = \psi_x(t; f) = f(x+t) - f(x-t)$.

2.4. Dini's test. *If the first of the integrals*

$$(1)\qquad \int_0^{\pi} \frac{|\varphi_x(t)|}{2\,\mathrm{tg}\,\frac{1}{2}t}\,dt,\quad \int_0^{\pi} \frac{|\psi_x(t)|}{2\,\mathrm{tg}\,\frac{1}{2}t}\,dt$$

is finite, then $\mathfrak{S}[f]$ *converges at* x *to the sum* $f(x)$. *If the second integral is finite,* $\overline{\mathfrak{S}}[f]$ *converges at the point* x *to the value which we shall denote by* $\bar{f}(x)$,

$$(2) \qquad \bar{f}(x) = -\frac{1}{\pi}\int_0^\pi \frac{\psi_x(t)}{2\,\mathrm{tg}\,\frac{1}{2}t}\,dt.$$

For the proof it is sufficient to observe that, in virtue of 2.3(4), the differences $s_n^*(x) - f(x)$ and $\bar{s}_n^*(x) - \bar{f}(x)$ are respectively the Fourier sine and cosine coefficients of integrable functions.

Since $2\,\mathrm{tg}\,\frac{1}{2}t \simeq t$ as $t \to 0$, the denominators in (1) may be replaced by t.

The integrals (1) converge if $\varphi_x(t) = O(t^\alpha)$, $\psi_x(t) = O(t^\alpha)$, $\alpha > 0$, as $t \to 0$; in particular if $f'(x)$ exists and is finite. However, the first of these integrals converges even when f is discontinuous at x, provided that $\frac{1}{2}\varphi_x(t) = \frac{1}{2}[f(x+t) + f(x-t)] - f(x)$ tends sufficiently rapidly to 0 with t. The second is divergent if only $f(x+0) \neq f(x-0)$ and, as we shall see later, $\overline{\mathfrak{S}}[f]$ will certainly diverge at such points.

If $f \in \mathrm{Lip}\,\alpha$, $\alpha > 0$, $\mathfrak{S}[f]$ *and* $\overline{\mathfrak{S}}[f]$ *converge everywhere.* It is easy to show that the convergence is uniform, but this theorem is contained in the more general result of § 2.71.

2.5. Theorems on localization. *If* f *vanishes in an interval* $I = (a, b)$, $\mathfrak{S}[f]$ *and* $\overline{\mathfrak{S}}[f]$ *converge uniformly in any interval* $I' = (a+\varepsilon, b-\varepsilon)$ *interior to* I, *and the sum of* $\mathfrak{S}[f]$ *is* 0 [1]. If the word 'uniformly' is omitted, the theorem becomes a simple corollary of Theorem 2.4, since, if $x \in I'$, $\varphi_x(t)$ and $\psi_x(t)$ vanish for small t and the integrals 2.4(1) are finite. We need the following lemma.

2.501. *Let* f *be integrable,* g *bounded* ($|g| < A$), *both periodic. The Fourier coefficients of the function* $\chi(t) = f(x+t)\,g(t)$, *depending on the parameter* x, *tend uniformly to* 0 [2].

It is sufficient to show that $\omega_1(\delta; \chi) \to 0$ with δ, uniformly in x. We have

---

1) Riemann [1], Lebesgue, *Leçons sur les séries trigonométriques*, 60, Hobson [1].

2) Hobson [1]; Plessner [1].

$$\int_{-\pi}^{\pi} |\chi(t+h)-\chi(t)|\,dt \leqslant \int_{-\pi}^{\pi} |f(x+t+h)-f(x+t)|\,|g(t+h)|\,dt$$
$$+\int_{-\pi}^{\pi} |f(x+t)|\,|g(t+h)-g(t)|\,dt.$$

If $|h| \leqslant \delta$, the first term on the right is less than $A\omega_1(\delta; f) \to 0$. To prove that the second term tends uniformly to 0, we put $|f| = f_1 + f_2$, where f_1 is bounded ($0 \leqslant f_1 < B$) and the integral of f_2 over $(-\pi, \pi)$ is less than $\varepsilon/4A$. The term considered is, obviously, less than $B\omega_1(\delta; g) + 2A \cdot \varepsilon/4A < \varepsilon$, for δ sufficiently small, and the lemma follows.

2.502. From the conditions of Theorem 2.5 we see that $f(x+t) = 0$ for $x \in I'$, $|t| < \varepsilon$. Let $\lambda(t)$ be equal to 0 for $|t| < \varepsilon$ and to 1 elsewhere. Using 2.3(3) we find that $s_n^*(x)$ is equal to

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\,\frac{\lambda(t)}{2\,\mathrm{tg}\,\frac{1}{2}t}\sin nt\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\,g(t)\sin nt\,dt,$$

where $g = \lambda/2\,\mathrm{tg}\,\frac{1}{2}t$. In virtue of Theorem 2.501, $s_n^*(x)$ tends uniformly to 0 if $x \in I'$. Similarly, if $\bar{f}(x)$ is given by 2.4(2), and $x \in I'$, $\bar{s}_n^*(x) - \bar{f}(x)$ tends uniformly to 0.

2.51. The results of the preceding paragraph may also be stated in a slightly different form. Two series $u_0 + u_1 + \ldots$ and $v_0 + v_1 + \ldots$ will be called *equiconvergent* if their difference $(u_0 - v_0) + (u_1 - v_1) + \ldots$ converges and has the sum 0 [1]. If the difference converges but not necessarily to 0, the series in question will be called *equiconvergent in the wider sense.*

If two functions f_1 and f_2 are equal in an interval I, then $\mathfrak{S}[f_1]$ and $\mathfrak{S}[f_2]$ are uniformly equiconvergent in any interval I' interior to I; $\bar{\mathfrak{S}}[f_1]$ and $\bar{\mathfrak{S}}[f_2]$ are uniformly equiconvergent in I' but in the wider sense.

For the proof we consider the difference $f = f_1 - f_2$.

Considering, for simplicity, convergence at a point, we may also put our results in the following form: The convergence of $\mathfrak{S}[f]$, $\bar{\mathfrak{S}}[f]$ and the sum of $\mathfrak{S}[f]$ (but *not* of $\bar{\mathfrak{S}}[f]$) at a point x, depend only on the behaviour of f in an arbitraily small neighbourhood of x. ('Riemann's principle of localization').

[1]) Szegö [1].

2.52. Approximate formulae for s_n. It is sometimes convenient to use the approximate formulae

$$(1)\qquad \begin{aligned} s_n(x) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\frac{\sin nt}{t}\,dt + o(1),\\ s_n(x) - f(x) &= \frac{1}{\pi}\int_0^{\pi} \varphi_x(t)\frac{\sin nt}{t}\,dt + o(1). \end{aligned}$$

In the first of them the error tends uniformly to 0, in the second it tends to 0 for every x, and uniformly in any interval where f is bounded. For the proof of the first result we observe that the difference of the integral on the right and the integral defining s_n^* is the Fourier coefficient of the function $f(x+t)\,g(t)$, where $g = 1/t - \frac{1}{2}\operatorname{ctg}\frac{1}{2}t$ is bounded in $(-\pi, \pi)$. In the second case we encounter the Fourier coefficients of the function equal to $[f(x+t) - f(x)]\,g(t)$ [1].

2.53. A theorem of Steinhaus [2]. *If at a point x_0 the derivates of a bounded function $\rho(x)$ are all finite, the series $\mathfrak{S}[\rho f]$ and $\rho(x_0)\,\mathfrak{S}[f]$ are equiconvergent at x_0.* In fact, the difference of the n-th partial sums of these series is equal to $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x_0+t)\,g(t)\sin nt\,dt$, where $g(t) = g_{x_0}(t) = [\rho(x_0+t) - \rho(x_0)]/2\sin\frac{1}{2}t$, and tends to 0, because it is the Fourier coefficient of an integrable function.

Suppose $\rho(x_0) = 1$. The theorem shows that 'slight' modifications of f in the neigbourhood of x_0 that leave $f(x_0)$ unaltered, have no influence either upon the convergence or the sum of $\mathfrak{S}[f]$ at x_0. More generally

2.531. *If $\rho(x)$ is periodic and satisfies the Lipschitz condition of order 1, the series $\mathfrak{S}[\rho f]$ and $\rho(x_0)\,\mathfrak{S}[f]$ are uniformly equiconvergent for all x_0. Similarly $\overline{\mathfrak{S}}[\rho f]$ and $\rho(x_0)\,\overline{\mathfrak{S}}[\rho f]$ are uniformly equiconvergent in the wider sense.*

We need only prove that $\omega_1(\delta; \chi) \to 0$ uniformly in x, where $\chi(t) = \chi_x(t) = f(x+t)\,g_x(t)$. Arguing as in the proof of Theorem

[1] If we replace $\sin nt$ by $\cos nt - 1$ in the first integral (1), we obtain an approximate expression for $\bar{s}_n(t)$, where the error tends uniformly to a continuous function.

[2] Steinhaus [1].

2.501, it remains to show, since $g_x(t)$ is uniformly bounded, $|g_x(t)| \leqslant M$, that $\int_{-\pi}^{\pi} |g_x(t+h) - g_x(t)|\, dt$ tends uniformly to 0 with h. Break up this integral into two, the first extended over $(-\varepsilon/8M, \varepsilon/8M)$. Since $g_x(t)$ is uniformly continuous outside this interval, the second integral tends uniformly to 0, and the first is less than $2.2M.\ \varepsilon/8M = \varepsilon/2$, so that the whole is less than ε, for h sufficiently small.

2.6. Functions of bounded variation. *If f is of bounded variation, $\mathfrak{S}[f]$ converges at every point x to the value $[f(x+0)+f(x-0)]/2$. If f is in addition continuous at every point of an interval $I=(a, b)$, $\mathfrak{S}[f]$ is uniformly convergent in I.* This theorem, due essentially to Dirichlet, is the first, chronologically, in the theory of Fourier series [1]). Its proof is elementary and uses only the results of § 2.213. We may suppose that at any point of simple discontinuity we have $f(x)=[f(x+0)+f(x-0)]/2$ [2]), so that the first part of the theorem asserts that $\mathfrak{S}[f]$ converges everywhere to $f(x)$. From 2.3(4) we have

$$(1) \quad s_n^*(x) - f(x) = \frac{1}{\pi}\left[\int_0^{\pi/n} + \int_{\pi/n}^{\eta} + \int_{\eta}^{\pi}\right] \frac{\varphi_x(t)}{2 \operatorname{tg} \frac{1}{2} t} \sin nt\, dt = P + Q + R,$$

where η will be defined in a moment. Since $|\sin nt| \leqslant nt \leqslant 2n \operatorname{tg} \frac{1}{2} t$, we see that $|P| \leqslant \operatorname{Max} |\varphi_x(t)|$ $(0 \leqslant t \leqslant 1/n)$ and so tends to 0. For fixed η, R is the Fourier coefficient of a function of bounded variation and hence is $O(1/n) = o(1)$. By the second mean-value theorem

$$(2) \quad Q = \tfrac{1}{2} \operatorname{ctg} \frac{\pi}{2n} \cdot \frac{1}{\pi} \int_{\pi/n}^{\eta'} \varphi_x(t) \sin nt\, dt, \quad \pi/n < \eta' < \eta.$$

Since $\varphi_x(t)$ is continuous for $t=0$, and $\varphi_x(0)=0$, the total variation

---

[1]) Dirichlet himself considered only functions having a finite number of maxima and minima, and in particular monotonic functions. Since, however, functions of bounded variation are differences of such functions, it is natural to associate Dirichlet's name with this theorem, which is only more general in appearance.

[2]) The set of points where a monotonic function, and so a function of bounded variation, is discontinuous, is at most enumerable.

of the function equal to $\varphi_x(t)$ in $(\pi/n, \eta')$ and to 0 elsewhere, is less than ε, if η is sufficiently small [1]). In virtue of Theorem 2.213, the second factor on the right in (2) is less than ε/n in absolute value, whence $|Q| < \varepsilon/\pi$. Therefore $|s_n^*(x) - f(x)| < o(1) + \varepsilon/\pi + o(1) < \varepsilon$ for $n > n_0$, i. e. $s_n^*(x) \to f(x)$.

If f is continuous in I, then, for $x \in I$, $\varphi_x(t)$ is uniformly small for small t and hence $P \to 0$ uniformly. For fixed η, the total variation of the function $\varphi_x(t)/2\,\mathrm{tg}\,\frac{1}{2}t$ over (η, π) is uniformly bounded [2]), and so again $R \to 0$ uniformly. If $x \in I$, $|t| < \delta$, the total variation of $f(x+t)$, and hence that of $\varphi_x(t)$, in a small intervall will be small [3]), and this gives as before that $|Q| < \varepsilon/\pi$ for η small but fixed. This completes the proof [4]).

2.601. A sequence of functions $f_n(x)$ convergent to $f(x)$ in a neighbourhood of a point x_0 is said to converge uniformly *at* the point x_0 if, for any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon)$ and a $p = p(\varepsilon)$ such that $|f(x) - f_n(x)| < \varepsilon$ for $|x - x_0| < \delta$, $n > p$.

If f is of bounded variation, $\mathfrak{S}[f]$ converges uniformly at every point x_0 where f is continuous. In fact, repeating the argument of § 2.6 it is easy to see that, if $|x - x_0|$ is small enough, the expression $|P| + |Q| + |R|$ is uniformly small.

2.61. Young's theorem. *If f is of bounded variation, a necessary and sufficient condition for the convergence of $\mathfrak{S}[f]$ at a point x is the existence of the integral*

$$(1)\qquad \overline{f}(x) = -\frac{1}{\pi}\int_0^\pi \frac{\psi_x(t)}{2\,\mathrm{tg}\,\frac{1}{2}t}\,dt = \lim_{h\to 0}\left[-\frac{1}{\pi}\int_h^\pi \frac{\psi_x(t)}{2\,\mathrm{tg}\,\frac{1}{2}t}\,dt\right],$$

---

[1]) The total variation of $\varphi(t)$ in an interval $\alpha \leqslant t \leqslant \alpha'$, $0 < \alpha < \alpha'$, tends to 0 if $\alpha' \to 0$, for otherwise there would exist a sequence of non-overlapping intervals (α_n, α_n') tending to 0, on which the total variation of φ would exceed a $\delta > 0$, and so φ would not be of bounded variation.

[2]) This follows e. g. from the obvious fact that if V_i, M_i denote respectively the total variation of g_i and $\mathrm{Max}|g_i|$, the total variation of $g_1 g_2$ is $\leqslant M_1 V_2 + M_2 V_1$.

[3]) The total variation is continuous wherever the function is continuous.

[4]) The decomposition of $s_n^* - f$ into *three* parts P, Q, R was not necessary, since it was not difficult to prove that $P + Q$ is small for small η (see the usual proof of Dirichlet's theorem in textbooks). However, the argument of the text can be applied to some other theorems.

which represents then the sum of $\bar{\mathfrak{S}}[f]$[1]. In virtue of Theorem 2.63 it is sufficient to consider only the points of continuity of f. Let $\bar{f}(x, h)$ denote the value of the integral (1) with the lower limit h instead of 0. Using the formula 2.3(4) we see that $\bar{s}_n^*(x) - \bar{f}(x, \pi/n)$ may be represented as the sum of three terms. Two of them are analogous to Q, R from the preceding section, with $\varphi_x(t) \sin nt$ replaced by $\psi_x(t) \cos nt$. The same proof as before shows that they tend to 0. The absolute value of the third is less than

$$\frac{1}{\pi}\int_0^{\pi/n} |\psi_x(t)| \frac{1-\cos nt}{2 \operatorname{tg} \frac{1}{2} t}\, dt \leqslant \frac{n^2}{2\pi}\int_0^{\pi/n} |\psi_x(t)|\, t\, dt = o(1).$$

It follows that $\bar{s}_n^*(x) - \bar{f}(x, \pi/n) \to 0$. In order to complete the proof it is enough to show that $\bar{f}(x, h) - \bar{f}(x, \pi/n) \to 0$ as $n \to \infty$, if $\pi/(n+1) < h < \pi/n$. But $|\bar{f}(x, h) - \bar{f}(x, \pi/n)| \leqslant [\pi/n - \pi/(n+1)] \cdot \frac{1}{2} \operatorname{ctg} \frac{1}{2} h \cdot \operatorname{Max} |\psi_x(t)| (0 < t \leqslant \pi/n) = o(1/n) = o(1)$.

2.62. Corollaries. *Let f be of bounded variation in an interval $I = (a, b)$. Then* (i) $\mathfrak{S}[f]$ *converges to* $[f(x+0)+f(x-0)]/2$ *at any point interior to I. If, besides that, f is continuous in I,* $\mathfrak{S}[f]$ *converges uniformly in every interval* $(a+\delta, b-\delta)$, (ii) *a necessary and sufficient condition for the convergence of* $\bar{\mathfrak{S}}[f]$ *at a point x interior to I, is the existence of the integral* 2.61(1), *which represents the sum of* $\bar{\mathfrak{S}}[f]$.

This follows immediately from Theorems 2.6, 2.61 and 2.51. Proposition (i) is known as 'Jordan's test'.

2.621. Integrated Fourier series. *Let F be the indefinite integral of f and let the first series in* 2.3(1) *be* $\mathfrak{S}[f]$. *Then we have, for* $-\infty < x < \infty$,

$$(1) \qquad F(x) = \frac{a_0 x}{2} + C + \sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx)/n,$$

the series on the right being uniformly convergent[2]. For the proof it is sufficient to observe that the series on the right, without its linear term, is the Fourier series of the function $F - a_0 x/2$, which is continuous and of bounded variation. If follows also that for every α, β we have

---

[1]) Young [2].

[2]) Lebesgue, *Leçons*, 102.

$\int\limits_\alpha^\beta f\,dx = \left[\frac{a_0 x}{2}\right]_\alpha^\beta + \sum\limits_{n=1}^{\infty}\left[\frac{a_n \sin nx - \cos nx}{n}\right]_\alpha^\beta$, i. e. Fourier series may be integrated term by term over any interval (α, β). From (1) we have:

If the first series in 2.3(1) *is a* $\mathfrak{S}[f]$, *the series* $b_1/1 + b_2/2 + \ldots$ *converges.* This may be false for the series $a_1/1 + a_2/2 + \ldots$ (See Chapter V).

2.622. *If f is of bounded variation, the partial sums of* $\mathfrak{S}[f]$ *are uniformly bounded.* We postpone the proof till § 3.23.

2.63. Conjugate series at points of discontinuity. We have seen that simple discontinuites are, in principle, no obstacles for the convergence of $\mathfrak{S}[f]$. For the conjugate series the situation is different: *If* $f(x+0) - f(x-0) = l > 0$, *then* $\overline{\mathfrak{S}}[f]$ *diverges at x to* $-\infty$ [1]).

This is contained in the following, more precise, result [2]).

2.631. *If* $f(x+0) - f(x-0) = l$, *then* $\bar{s}_n(x)/\log n \to -l/\pi$.

Since $f(x+t) - f(x-t) = l + \varepsilon(t)$, $\varepsilon(t) \to 0$, we may write

$$(1) \qquad \bar{s}_n^*(x) = -\frac{l}{\pi}\int\limits_0^\pi \bar{D}_n^*(t)\,dt - \frac{1}{\pi}\int\limits_0^\pi \varepsilon(t)\,\bar{D}_n^*(t)\,dt.$$

To find the first of the integrals on the right, let us denote them by I_n, I_n' and consider the function $f(t) = (\pi - t)/2$ $(0 < t < 2\pi)$. Here $l = f(+0) - f(-0) = \pi$, $\varepsilon(t) = -t$, $\bar{s}_n^*(0) = -1 - 1/2 - \ldots - 1/(n-1) - 1/2n = -\log n + O(1)$. Substituting this in (1) we find that $I_n = \log n + O(1) \simeq \log n$. Now we will show that $I_n' = o(\log n)$. We break up this integral into two, the first of which is extended over $(0, \delta)$, where δ is so small that $|\varepsilon(t)| < \eta/2$ for $0 \leqslant t \leqslant \delta$. Since $\bar{D}_n^* \geqslant 0$, the first term is less than $\eta I_n/2$. The second term is bounded, and so less than $\eta I_n/2$ in absolute value for $n > n_0$. It follows that $|I_n'| < \eta I_n$ $(n > n_0)$, i. e. $I_n' = o(I_n) = o(\log n)$. This completes the proof.

This theorem gives us a means of determining the simple discontinuities of functions from their Fourier series.

2.632. Corollaries. (i) If the Fourier coefficients of a function f are $o(1/n)$, f cannot possess simple discontinuities. In

[1]) Pringsheim [1].

[2]) Lukács [1].

fact, for such functions $\bar{s}_n^*(x) = o(\log n)$ uniformly in x (§§ 1.72, 1.74). In particular, *if the Fourier coefficients of a function f of bounded variation are $o(1/n)$, f is continuous.*

(ii) If f is continuous at a point x, then $\bar{s}_n(x) = o(\log n)$. If f is continuous in an interval (a, b), then $\bar{s}_n(x) = o(\log n)$, uniformly in every interval $(a+\delta, b-\delta)$.

2.7. Lebesgue's test. Let $\varphi(t) = \varphi_x(t)$, $\chi(t) = \varphi(t)/2 \operatorname{tg} \frac{1}{2}t$, $\eta = \pi/n$.

We begin by proving the following lemma.

2.701. *For every x, $\pi|s_n^*(x) - f(x)|$ is less than*

$$(1)\quad \int_\eta^\pi \frac{|\varphi(t) - \varphi(t+\eta)|}{t}\,dt + A\eta \int_\eta^\pi \frac{|\varphi(t)|}{t^2}\,dt + 2n\int_0^{2\eta} |\varphi(t)|\,dt + o(1),$$

where A is an absolute constant. The last term on the right tends to 0 uniformly in any interval where f is bounded. Applying the device of § 2.21, we see that $\pi[s_n^*(x) - f(x)]$ is equal to

$$\int_0^\pi \chi(t)\sin nt\,dt - \int_{-\eta}^{\pi-\eta} \chi(t+\eta)\sin nt\,dt = \int_\eta^{\pi-\eta} [\chi(t) - \chi(t+\eta)]\sin nt\,dt +$$

$$+ \int_{\pi-\eta}^{\pi} \chi(t)\sin nt\,dt + \int_0^\eta \chi(t)\sin nt\,dt + \int_0^{2\eta} \chi(t)\sin nt\,dt.$$

Let us denote the integrals on the right by I_1, I_2, I_3, I_4 respectively. The sum $|I_3| + |I_4|$ is less than the third term in (1). We may assume that $|\chi(t)\sin nt| \leqslant |\varphi(t)| \leqslant |f(x+t)| + |f(x-t)| + |2f(x)|$ for $t \in (\pi-\eta, \pi)$ and, since an indefinite integral is a continuous function, we see that $I_2 \to 0$. Finally, $|I_1|$ is less than

$$\int_\eta^{\pi-\eta} \frac{|\varphi(t) - \varphi(t+\eta)|}{2\operatorname{tg}\frac{1}{2}(t+\eta)}\,dt + \int_\eta^{\pi-\eta} |\varphi(t)| \left[\frac{1}{2\operatorname{tg}\frac{1}{2}t} - \frac{1}{2\operatorname{tg}\frac{1}{2}(t+\eta)}\right] dt.$$

The difference in square brackets is equal to the expression $\sin\frac{1}{2}\eta/\sin\frac{1}{2}t\,\sin\frac{1}{2}(t+\eta) \leqslant A\eta/t^2$.

This completes the proof.

2.702. Let $\Phi(h) = \Phi_x(h)$ be the integral of $|\varphi_x(t)|$ over $(0, h)$. Lebesgue's test may be formulated as follows: $\mathfrak{S}[f]$ *converges to the value $f(x)$ at every point x at which*

$$(1) \qquad \Phi_x(h) = o\,(h), \quad \int_\eta^\pi \frac{|\varphi(t) - \varphi(t+\eta)|}{t}{}^{1)}\,dt \to 0$$

as $\eta \to 0$. Using Lemma 2.701, it remains to show that the second term in 2.701(1) is $o\,(1)$. Integrating by parts, we find for it the value $A\eta\,\{[\Phi(t)\,t^{-2}]_\eta^\pi + 2\int_\eta^\pi \Phi(t)\,t^{-3}\,dt\} = o\,(1)$, since $\Phi(t) = o\,(t)$, (§ 1.71).

2.703. An important discovery of Lebesgue is that the first condition in 2.702(1) is satisfied almost everywhere. The result may be stated in the following form.

Let $F_x(h) = \int_0^h |f(x+t) - f(x)|\,dt$. *Then, for almost every* x, *we have* $F_x(h) = o\,(h)$ *as* $h \to \pm 0$. This proposition represents a generalization of the well-known theorem on the differentiability of an integral, to which it reduces if we omit the sign of absolute value in the definition of F. Les us denote by E_α, where α is rational, the set of x for which the relation $\frac{1}{h}\int_0^h |f(x+t) - \alpha|\,dt \to |f(x) - \alpha|$ does not hold. In virtue of the theorem just mentionned, any E_α is of mesure 0, and so the sum E of all E_α is of mesure 0. We will prove that $F_x(h) = o\,(h)$ for $x \bar{\epsilon} E$. Suppose that $\varepsilon > 0$ is given and let β be a rational number such that $|f(x) - \beta| < \varepsilon/2$. In the inequality

$$F_x(h) \leqslant \int_0^h |f(x+t) - \beta|\,dt + \int_0^h |\beta - f(x)|\,dt,$$

where, for simplicity, $h > 0$, the ratio of the first integral on the right to h tends to $|f(x) - \beta| < \varepsilon/2$. Hence, for small h, we have $F_x(h) < \varepsilon\,h/2 + \varepsilon\,h/2 = \varepsilon\,h$, and, ε being arbitrary, the result follows.

2.71. The Dini-Lipschitz test. *If f is continuous and its modulus of continuity satisfies the condition* $\omega(\delta)\log 1/\delta \to 0$, *as* $\delta \to 0$, *then* $\mathfrak{S}[f]$ *converges uniformly*. This follows from Lemma 2.701. Since $|\varphi(t) - \varphi(t+\eta)| \leqslant |f(x+t) - f(x+t+\eta)| + |f(x-t) - f(x-t-\eta)| < 2\omega(\eta)$,

---

[1]) The upper limit of integration π may be replaced by any fixed $\sigma > 0$ (§ 2.201).

the first term in 2.701(1) is $\leqslant 2\omega(\eta)\log\pi/\eta \to 0$. Similarly, since $\varphi(t)\to 0$ uniformly in x, the remaining terms in 2.701(1) tend uniformly to 0 (§ 1.71).

The result holds in particular if $f \in \operatorname{Lip}\alpha$ $(\alpha > 0)$.

In virtue of the theorems on localization, we conclude that *if f is continuous in an interval $I = (a, b)$ and if the modulus of continuity of f in this interval is $o(\log 1/\delta)^{-1}$, $\mathfrak{S}[f]$ converges uniformly in every interval $(a+\varepsilon, b-\varepsilon)$.* This test is known as the Dini-Lipschitz test and is primarily a condition for uniform convergence. We shall see in Chapter VIII that the condition $f(x_0+h) - f(x_0) = o(\log 1/|h|)^{-1}$ does not ensure the convergence of $\mathfrak{S}[f]$ at x_0.

2.72. In the preceding section we proved that, if in an interval (a, b) the function f satisfies a Lipschitz condition of positive order, then $\mathfrak{S}[f]$ and $\bar{\mathfrak{S}}[f]$ converge uniformly in every interval $(a+\varepsilon, b-\varepsilon)$. We will now prove a slightly more precise result, which completes that established in § 2.4.

If $f(x) \in \operatorname{Lip}\alpha$, $\alpha > 0$, in an interval (a, b), and if, moreover, $|f(b+t) - f(b)| < At^\alpha$, $|f(a) - f(a-t)| < At^\alpha$, $0 < t \leqslant h$, where A is a constant, then $\mathfrak{S}[f]$ and $\bar{\mathfrak{S}}[f]$ converge uniformly in (a, b) [1]. There exists a constant $B > 0$, such that $|f(x+t) - f(x)| \leqslant Bt^\alpha$, if only $a \leqslant x \leqslant b$, $|t| \leqslant h$, and so, in the equation

$$s_n^*(x) - f(x) = \frac{1}{\pi}\left\{\int_{-\sigma}^{\sigma} + \left(\int_{-\pi}^{-\sigma} + \int_{\sigma}^{\pi}\right)\right\}[f(x+t) - f(x)]\, D_n^*(t)\, dt = P + Q,$$

where $0 < \sigma \leqslant h$, the integrand of P does not exceed $B|t|^{\alpha-1}$ in absolute value. Hence, taking σ small enough, we have $|P| < \varepsilon/2$, uniformly in (a, b). Since Q is the Fourier coefficient of the function $[f(x+t) - f(x)]\, g(t)$, where $g(t) = \frac{1}{2}\operatorname{ctg}\frac{1}{2}t$ for $\sigma \leqslant |t| \leqslant \pi$, $g(t) = 0$ for $|t| < \sigma$, we see, by Theorem 2.501, that $Q \to 0$ uniformly as $n \to \infty$, so that $|Q| < \varepsilon/2$, $|P+Q| < \varepsilon$, for $n > n_0$, $a \leqslant x \leqslant b$. In the same way we prove the uniform convergence of $\bar{s}_n^*(x) - \bar{f}(x)$.

Let $\omega(\delta)$ denote the modulus of continuity of f in the interval (a, b). If $\omega(\delta)/\delta$ is integrable in a neighbourhood of $\delta = 0$, and if $|f(b+t) - f(b)| < A\omega(t)$, $|f(a) - f(a-t)| < A\omega(t)$, $0 < t \leqslant h$, then $\mathfrak{S}[f]$ and $\bar{\mathfrak{S}}[f]$ converge uniformly in (a, b). The proof re-

[1]) Hobson, *Theory of functions*, 2, 535.

mains the same as above. The result holds, in particular, if $\omega(\delta)=O(\log 1/\delta)^{-1-\varepsilon}$, $\varepsilon>0$. For $\varepsilon=0$ the argument fails, and, as we shall see later, the theorem itself is false.

2.73. As we shall see in Chapter VIII, the partial sums of $\mathfrak{S}[f]$ may be unbounded almost everywhere. However

If at a point x, we have $\Phi_x(h)=o(h)$, then $s_n(x)=o(\log n)$, and, if $\Psi_x(h)=o(h)$, then $\bar{s}_n(x)=o(\log n)$ [1]. We know that $\Phi_x(h)=o(h)$, $\Psi_x(h)=o(h)$ almost everywhere. From 2.3(4) we see that the expression $\pi|s_n^*(x)-f(x)|$ does not exceed

$$n\int_0^{1/n}|\varphi(t)|\,dt+\int_{1/n}^{\pi}t^{-1}|\varphi(t)|\,dt=n\Phi(1/n)+[\Phi(t)\,t^{-1}]_{1/n}^{\pi}+\int_{1/n}^{\pi}\Phi(t)\,t^{-2}\,dt.$$

The first two terms on the right give $\Phi(\pi)/\pi=O(1)=o(\log n)$, the third, in virtue of the relation $\Phi(t)=o(t)$, is $o(\log n)$ (§ 1.71). We proceed similarly with $|\bar{s}_n^*(x)|$, taking into account that $|\bar{D}_n^*(t)|\leqslant n$ for $0\leqslant t\leqslant 1/n$, and $|\bar{D}_n^*(t)|\leqslant 2/t$ if $1/n\leqslant t\leqslant\pi$.

If f is continuous in (a,b), then $s_n(x)/\log n$ and $\bar{s}_n(x)/\log n$ tend uniformly to 0 for $x\in(a+\delta,b-\delta)$ $(\delta>0)$. The proof is still simpler since in the inequalities for $|\bar{s}_n^*|$ and $|s_n^*-f|$ no integration by parts is necessary.

2.74. Lebesgue's criterion has an analogue for conjugate series. Let $\Psi_x(h)$ be the integral of $|\psi_x(t)|$ over $(0,h)$ and let $\bar{f}(x,h)$ have the same meaning as in § 2.61. Then, *the conditions*

$$(1)\qquad \Psi_x(h)=o(h),\quad \int_{\eta}^{\pi}\frac{|\psi(t)-\psi(t+\eta)|}{t}\,dt\to 0\quad (h,\eta\to 0)$$

involve the relation $\bar{s}_n^*(x)-\bar{f}(x,\pi/n)\to 0$. In other words, under the above conditions, $\bar{\mathfrak{S}}[f]$ converges at a point x if and only if the integral 2.61(1) exists [2]. The conditions (1) will certainly be satisfied if f satisfies the Dini-Lipschitz condition in an interval containg x. The proof we leave to the reader.

If $f\in\text{Lip}\,\alpha$, then $\bar{\mathfrak{S}}[f]$ converges uniformly. This follows from the fact that $\bar{s}_n^*(x)-\bar{f}(x,\pi/n)$ tends uniformly to 0 and that the integral $\bar{f}(x,\eta)$ converges uniformly.

---

1) Hardy [2]; Young [3].

2) If $\pi/(n+1)<h\leqslant\pi/n$, then $|\bar{f}(x,h)-\bar{f}(x,\pi/n)|\leqslant(n+1)\Psi_x(\pi/n)/\pi^2\to 0$.

2.8. de la Vallée Poussin's test. *If the function* $\chi(t) = \chi_x(t) = \frac{1}{t}\int_0^t \varphi_x(u)\,du$ *is of bounded variation in an interval to the right of* $t=0$, *and if* $\chi(t)\to 0$ *as* $t\to 0$, *then* $\mathfrak{S}[f]$ *converges at* x *to the value* $f(x)$ [1].

The convergence of $\mathfrak{S}[f]$ at x to $f(x)$ is the same thing as the convergence of $\mathfrak{S}[\varphi]$ at the point $t=0$ to the value 0. Now $\varphi(t) = t\chi'(t) + \chi(t)$ and, since the derivative of a function of bounded variation is integrable, φ is the sum of two functions, the first of which satisfies Dini's condition at $t=0$ and the second is of bounded variation.

2.81. Young's test [2]. $\mathfrak{S}[f]$ *converges at the point* x *to the value* $f(x)$, *provided that* (1) $\varphi_x(t)\to 0$ *as* $t\to 0$, (2) *the function* $\theta(t) = t\varphi_x(t)$ *is of bounded variation in an interval to the right of* $t=0$, *and* (3) *the total variation* $v(h)$ *of* θ *over* $(0,h)$ *is* $\leqslant Ah$ *for small* h, *where* A *is a constant.*

Consider the decomposition of the integral 2.52(1) defining $s_n - f$ into three integrals P, Q, R, extended over the intervals $(0, k/n)$, $(k/n, \eta)$, (η, π), where k is large but fixed, and η is defined by the condition that θ is of bounded variation in $(0,\eta)$. We have $|P| \leqslant n\Phi_x(k/n)\to 0$. Similarly $R\to 0$. Q is the sine coefficient of a function $\xi(t) = \xi_n(t)$ of bounded variation, and the theorem will have been proved when we have shown that the total variation of ξ over $(0,\pi)$ is less than εn (ε arbitrary >0), if only k is made large enough [3]. Since $\xi(k/n) = o(n)$, $\xi(\eta) = O(1)$, $\xi(t) = 0$ outside $(k/n, \eta)$, it is enough to prove the same thing for the variation of $\xi(t) = \theta(t)/t^2$ over the interior of $(k/n, \eta)$.

Let $(a,b) = (k/n, \eta)$, $\alpha(t) = t^{-2}$, $\beta(t) = \theta(t)$, $u(t) =$ the total variation of α over (a,t), $v(t) =$ the total variation of β over (a,t), and let $a = t_0 < t_1 < \ldots < t_m = b$ be any subdivision of (a,b). If we add the obvious inequalities $|\alpha(t_i)\beta(t_i) - \alpha(t_{i-1})\beta(t_{i-1})| \leqslant \leqslant |\alpha(t_i)|\,|\beta(t_i)-\beta(t_{i-1})| + |\beta(t_{i-1})|\,|\alpha(t_i)-\alpha(t_{i-1})| \leqslant |\alpha(t_i)|[v(t_i) - - v(t_{i-1})] + |\beta(t_{i-1})|[u(t_i) - u(t_{i-1})]$, $i = 1, 2, \ldots, m$, we find that the total variation of $\alpha\beta$ over (a,b) does not exceed

---

[1]) de la Vallée Poussin [1].
[2]) Young [4]; Hardy and Littlewood [1].
[3]) The argument is similar to that used in § 2.6.

$$\int_a^b |\alpha(t)|\,dv(t) + \int_a^b |\beta(t)|\,du(t) = \int_{k/n}^{\eta} t^{-2}\,dv(t) + 2\int_{k/n}^{\eta} |\theta(t)|\,t^{-3}\,dt.$$

Since $|\theta(t)| = |\theta(t) - \theta(0)| \leqslant v(t) \leqslant At$, the last integral is less than $2A\,n/k$. An integration by parts shows the preceding integral to be less than $[v(\eta)\eta^{-2} - v(k/n)(k/n)^{-2}] + 2A\,n/k$. Altogether the two integrals yield less than $O(1) + 4\,An/k < \varepsilon n$, if k is large enough.

2.82. The following theorem, in which $\bar{f}(x, \eta)$ has the same meaning as in § 2.61, is an extension to the case of conjugate series of the results proved in §§ 2.8, 2.81.

The difference $\bar{s}_n(x) - \bar{f}(x, \pi/n)$ tends to 0 as $n \to \infty$, if one of the following two conditions is satisfied:[1]

(i) *the function $\chi(t) = \dfrac{1}{t}\displaystyle\int_0^t \psi_x(u)\,du$ is of bounded variation in an interval to the right of $t = 0$.*

(ii) *$\psi_x(t) \to 0$ with t, $t\psi(t)$ is of bounded variation in an interval to the right of $t = 0$, and the total variation of $t\psi(t)$ over $(0, h)$ is $O(h)$.*

To prove the first part of the theorem we observe that $\bar{\mathfrak{S}}[f]$ at the point x is the same thing as $\frac{1}{2}\bar{\mathfrak{S}}[\psi]$ at $t = 0$. Now $\psi(t) = \psi_1(t) + \psi_2(t)$, $\psi_1(t) = t\,\chi'(t)$, $\psi_2(t) = \chi(t)$ and so we have $2[\bar{s}_n(x; f) - \bar{f}(x, \pi/n)] = \bar{s}_n(0; \psi) - \bar{\psi}(0, \pi/n) = [\bar{s}_n(0; \psi_1) - \bar{\psi}_1(0; \pi/n)] + [\bar{s}_n(0; \psi_2) - \bar{\psi}_2(0, \pi/n)]$.

Since $\bar{\psi}_1(0, \pi/n) \to \bar{\psi}_1(0)$, $\bar{s}_n(0; \psi_1) \to \bar{\psi}_1(0)$ (§ 2.4), $\bar{s}_n(0; \psi_2) - \bar{\psi}_2(0, \pi/n) \to 0$ (§ 2.61), we obtain that $\bar{s}_n(x) - \bar{f}(x, \pi/n) \to 0$ and this gives the first part of the theorem.

The proof of the second part is much the same as that of Theorem 2.81.

2.83. The Hardy-Littlewood test. This test is interesting because it takes into account not only the behaviour of the function, but also that of the Fourier coefficients.

[1] Young [2], [5]

$\mathfrak{S}[f]$ converges at the point x to the value $f(x)$, if the following two conditions are satisfied (i) $f(x+h)-f(x)=o\,(\log 1/|h|)^{-1}$, (ii) *the coefficients of* $\mathfrak{S}[f]$ *are* $O(n^{-\delta})$, $\delta>0$[1].

Since instead of $\mathfrak{S}[f]$ we may consider $\mathfrak{S}[\varphi]$, let us assume that $x=0$, $f(0)=0$, $f(x)=f(-x)$, $|a_n|<n^{-\delta}$, $0<\delta<1$. It is also convenient to suppose that $a_0=0$[2]. Let $r=\delta/2$. We have

$$s_n^*(0)=\frac{2}{\pi}\int_0^{\pi} f(t)\frac{\sin nt}{2\,\mathrm{tg}\,\frac{1}{2}t}\,dt=\int_0^{n^{-1}}+\int_{n^{-1}}^{n^{-r}}+\int_{n^{-r}}^{\pi}=P+Q+R.$$

Since f is continuous at the point 0, $P\to 0$ as $n\to\infty$. If $\varepsilon(t)=$ $=\operatorname{Max}\{|f(u)|\log 1/u\}$ for $0<u\leqslant t$, then

$$|Q|\leqslant \varepsilon(n^{-r})\int_{n^{-1}}^{n^{-r}}\frac{dt}{t\log 1/t}=\varepsilon(n^{-r})\log 1/r\to 0,$$

and it remains only to show that $R\to 0$. Using the theorem (which will be established in Chapter IV) that Fourier series may be integrated term by term after having been multiplied by an arbitrary function of bounded variation, we have

$$R=\sum_{k=1}^{\infty}a_k\frac{2}{\pi}\int_{n^{-r}}^{\pi}\frac{\sin nt\cos kt}{2\,\mathrm{tg}\,\frac{1}{2}t}\,dt.$$

Replacing the products $\cos kt\sin nt$ by differences of sines, and applying the second mean-value theorem to the factor $\frac{1}{2}\,\mathrm{ctg}\,\frac{1}{2}t$, we see that the coefficient of a_k, $k\neq n$, does not exceed $4n^r/\pi\,|k-n|$ in absolute value, and so

$$|R|\leqslant o(1)+\frac{4}{\pi}\sum_{k=1}^{\infty}{}'\frac{k^{-\delta}n^r}{|k-n|}=o(1)+\sum_{k=1}^{n-1}+\sum_{k=n+1}^{\infty}=o(1)+R_1+R_2,$$

where $'$ denotes that the term $k=n$ is omitted. Now

$$\frac{\pi}{4}R_1<\frac{n^r}{\frac{1}{2}n}\sum_{k=1}^{[\frac{1}{2}n]}k^{-\delta}+n^r(\tfrac{1}{2}n)^{-\delta}\sum_{k=[\frac{1}{2}n]+1}^{n-1}\frac{1}{n-k}=O(n^{-\frac{\delta}{2}})+O(n^{-\frac{\delta}{2}}\log n)=o(1)$$

$$\frac{\pi}{4}R_2<n^{r-\delta}\sum_{k=n+1}^{2n}\frac{1}{k-n}+n^r\sum_{k=2n+1}^{\infty}\frac{k^{-\delta}}{\frac{1}{2}k}=O(n^{-\frac{\delta}{2}}\log n)+O(n^{-\frac{\delta}{2}})=o(1),$$

and this completes the proof. The same argument shows that

Under the conditions of the above theorem, $\bar{s}_n(x)-\bar{f}(x,\pi/n)\to 0$.

---

[1]) Hardy and Littlewood [2], [3].

[2]) We can secure this by adding $-\frac{1}{2}a_0(1-\cos x)$ to $\mathfrak{S}[f]$.

2.84. Relations between tests [1]. We shall consider only tests for the convergence of Fourier series.

Dini's and Jordan's tests are not comparable [2]. Let $f(x)$, $g(x)$ be even and let $f(x) = 1/\log(x/2\pi)$, $g(x) = x^\alpha \sin 1/x$ $(0 < \alpha < 1)$ for $0 < x \leqslant \pi$. At the point 0, f satisfies Jordan's condition but not Dini's, and conversely g satisfies Dini's condition but not Jordan's.

de la Vallée Poussin's test includes both Dini's and Jordan's. Let $\Phi(t)$ be the integral of φ over $(0, t)$, and let $\chi(t) = \Phi(t)/t$. If φ is positive and non-decreasing, so is χ. If φ is of bounded variation, i. e. if $\varphi = \varphi_1 - \varphi_2$, where φ_1, φ_2 are positive and non-decreasing, then $\chi = \chi_1 - \chi_2$ is also of bounded variation. This proves the second part of the theorem. To prove the first, let $\mu(t)$ be the integral of $\varphi(u)/u$ over $(0, t)$. A simple calculation shows that

$$\frac{1}{t}\int_0^t \varphi(u)\,du = \mu(t) - \frac{1}{t}\int_0^t \mu(u)\,du,$$

and if μ is of bounded variation the same is true for the expression on the left.

de la Vallée Poussin's and Young's tests are not comparable. Let $g(x)$ be even, $g(x) = (-1)^n x^\alpha$ for $\pi/(n+1) < x \leqslant \pi/n$, $n = 1, 2, \ldots$ The total variation of $xg(x)$ over $(0, \pi/n)$ is exactly of order $n^{-\alpha}$. It follows that, if $0 < \alpha < 1$, $x = 0$, g satisfies Dini's condition but not Young's. Thus Young's condition does not include Dini's, and, à fortiori, de la Vallée Poussin's.

Let $h(x)$ be even and equal to $(-1)^n \beta_n$ in the interval $(\pi 2^{-n-1}, \pi 2^{-n})$, $n = 0, 1, 2, \ldots$, where $1 > \beta_0 > \beta_1 > \ldots \to 0$. A simple calculation shows that the total variation of $H(x) = x^{-1}\int_0^x h(t)\,dt$ over $(\pi 2^{-n-1}, \pi 2^{-n})$ is equal to $[\beta_n/2 + \beta_{n+1}/2^2 - \beta_{n+2}/2^3 + \beta_{n+3}/2^4 - \ldots] > \beta_n/2$, so that, if $\beta_1 + \beta_2 + \ldots = \infty$, $h(x)$ does not satisfy de la Vallée Poussin's condition at the point $x = 0$. From the graph of the curve $y = \theta(x) = xh(x)$ we deduce that, if $\pi 2^{-n-1} \leqslant x < \pi 2^{-n}$, the total

---

[1]) Hardy [3]. See also Gergen [1], Pollard [1].

[2]) We say that f satisfies Jordan's condition at a point x_0, if $f(x)$ is of bounded variation in a neighbourhood of x_0 (§ 2.62).

variation of θ over $(0, x)$ is less than $o(x)+2\pi[\beta_n 2^{-n-1}+\beta_{n+1}2^{-n-2}+ + \dots] \leqslant o(x)+\beta_n \pi 2^{1-n} = o(x)$, and so Young's condition is satisfied.

We state without proof the following result: *de la Vallée Poussin's and Young's tests are both included in Lebesgue's test* [1]), which, consequently, turns out to be the most powerful, although not always the most convenient, of all the tests discussed in this section.

2.85. Poisson's formula. Let $g(x)$ be a function defined for $-\infty < x < \infty$, tending to 0 as $x \to \pm\infty$, and integrable in any finite interval. Suppose, moreover, that the series

$$(1) \qquad \sum_{k=-\infty}^{+\infty} g(k+x) = G(x),$$

whose symmetric partial sums we denote by $G_n(x)$, converges uniformly [2]) for $0 \leqslant x \leqslant 1$. The sum $G(x)$ is of period 1, and its Fourier coefficients c_ν with respect to the system $\{\exp 2\pi i\nu x\}$ are

$$(2) \qquad \lim_{N\to\infty}\int_0^1 G_N\, e^{-2\pi i\nu x}\,dx = \lim_{N\to\infty}\int_{-N}^{N+1} g e^{-2\pi i\nu x}\,dx = \int_{-\infty}^{+\infty} g(x)\, e^{-2\pi i\nu x}\,dx.$$

Hence, supposing that, at the point $x = 0$, G satisfies one of the conditions ensuring the convergence of $\mathfrak{S}[G]$ to the value $G(0)$, we obtain immediately the Poisson formula

$$(3) \qquad \sum_{k=-\infty}^{+\infty} g(k) = \sum_{\nu=-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x)\, e^{-2\pi i\nu x}\,dx.$$

This formula is true if, for example, g is of bounded variation over $(-\infty, +\infty)$, $2g(x) = g(x+0)+g(x-0)$, and if the series (1) converges at a point. In fact, let v_k be the total variation of $g(x)$ over $(k, k+1)$. Since the oscillation of $g(x+k)$ in $(0,1)$ does not exceed v_k, and $\dots + v_{-1} + v_0 + v_1 + \dots = V < \infty$, the series in (1) converges uniformly. $G(x)$ is of bounded variation since its total variation over $(0, 1)$ does not exceed V. Moreover, it is easy to see that $2G(x) = G(x+0)+G(x-0)$.

An additional remark on the Fourier coefficients of the function $G(x)$ in (1) will be useful later. It may happen that $G_n(x)$

---

[1]) H a r d y [3]; H o b s o n, *Theory of functions;* 2, 533.

[2]) This condition might be relaxed.

itself does not tend to any limit, but that there exists a sequence of constants K_n such that the sequence $H_n(x) = G_n(x) - K_n$ does tend, uniformly, to a limit $H(x)$. Changing, if necessary, the values of K_n, we may suppose that the integral of H over $(0, 1)$ vanishes, so that, if now c_ν are the complex Fourier coefficients of H, we have $c_0 = 0$. Taking into account that the integral of $K_n \exp(-2\pi i\nu x)$ over $(0, 1)$ vanishes, and replacing in (2) G_N by H_N, we find the same formula as before for c_ν. In other words, since K_n may be taken as the mean-value of G_n over $(0, 1)$, we may write

$$(4) \qquad \lim_{n\to\infty}\left\{G_n(x) - \int_0^1 G_n(t)\,dt\right\} \sim \sum_{\nu=-\infty}^{+\infty}{}' e^{2\pi i\nu x}\int_{-\infty}^{+\infty} g(t)\, e^{-2\pi i\nu t}\,dt,$$

where $'$ denotes that the term $\nu = 0$ is omitted.

Example. Let $g(x) = x^{-\alpha}$ for $x > 0$, $g(x) = 0$ elsewhere, $0 < \alpha < 1$. Here $G_n(x) = x^{-\alpha} + (x+1)^{-\alpha} + \ldots + (x+n)^{-\alpha}$, $K_n = (n+1)^{1-\alpha}/(1-\alpha)$. Therefore, since $(n+1)^{1-\alpha} - n^{1-\alpha} \to 0$, the numbers $c_\nu = \int_0^\infty x^{-\alpha} e^{2\pi i\nu x}\,dx$ are the Fourier coefficients of the function

$$\lim_{n\to\infty}\left[x^{-\alpha} + (x+1)^{-\alpha} + \ldots + (x+n)^{-\alpha} - n^{1-\alpha}/(1-\alpha)\right] \quad (0 < x < 1).$$

2.9. Miscellaneous theorems and examples.

1. If $\omega_1(\delta; f) = o(\delta)$, then $f \equiv \text{const}$. Titchmarsh, *Theory of functions*, 372.

[Consider $\int_{x_1}^{x_2} [f(t+h) - f(t)]\,dt$].

2. Given an arbitrary sequence $\varepsilon_n \to 0$, $\varepsilon_n > 0$, there exists a continuous f such that $|a_n| + |b_n| \geqslant \varepsilon_n$ for infinitely many n. Lebesgue [1].

[If $n_1 < n_2 < \ldots$ and $\varepsilon_{n_1} + \varepsilon_{n_2} + \ldots < \infty$, put $f(x) = \varepsilon_{n_1}\cos n_1 x + \varepsilon_{n_2}\cos n_2 x + \ldots$].

3. Let $f(x) = a\cos bx + a^2\cos b^2 x + \ldots + a^n \cos b^n x + \ldots$, $0 < a < 1$, $ab > 1$. Show that (i) $f \in \mathrm{Lip}\,\alpha$, where $\alpha = \log a^{-1}/\log b$, (ii) the Fourier coefficients of f are $O(n^{-\alpha})$, but not $o(n^{-\alpha})$ (iii) if $ab = 1$, then $\omega(\delta; f) = O(\delta \log 1/\delta)$. Hardy [4].

[Let $\nu = \nu(h)$ be the largest n such that $b^n h \leqslant 1$. In the formula

$$f(x+h) - f(x-h) = -\sum_{n=1}^{\infty} 2a^n \sin b^n h \sin b^n x = \sum_{\nu=1}^{\nu} + \sum_{n=\nu+1}^{\infty} = P + Q$$

the terms of P do not exceed $2a^n b^n h$, so that $P = O(h^\alpha)$. The terms of Q are $\leqslant 2a^n$, and so $Q = O(h^\alpha)$].

4. Using Theorem 2.622 and the equation $\sum_{\lambda}^{\mu}(a_n \sin nx - b_n \cos nx)/n = = \frac{1}{\pi}\int_0^{2\pi} f(t) \sum_{\lambda}^{\mu} \frac{\sin n(x-t)}{n}\,dt$, prove Theorem 2.621 and the formula

$$\sum_{n=1}^{\infty} \frac{b_n}{n} = \frac{1}{\pi}\int_0^{2\pi} f(t)\,\frac{\pi - t}{2}\,dt.$$

5. The numbers $C_k = 1 + 2^{-2k} + 3^{-2k} + \ldots + n^{-2k} + \ldots$, $k = 1, 2, 3, \ldots$, are all rational multiples of π^{2k}.

[Integrate the series $\sin x + \frac{1}{2}\sin 2x + \ldots$ an odd number of times].

6. If $f(x)$ has k derivatives, the Fourier coefficients of f satisfy the relation $|c_n| \leqslant \omega(\pi/n; f^{(k)})/2n^k$, $n > 0$. If $f^{(k)}$ is of bounded variation, then $c_n = O(n^{-k-1})$.

7. If $f(x)$ vanishes in (a, b), the function $\overline{f}(x)$ defined by 2.4(2) has derivatives of any order for $a < x < b$.

8. Considering $\mathfrak{S}[\cos \alpha x]$, prove the formulae

$$\frac{\alpha\pi}{\sin \alpha\pi} = 1 + 2\alpha^2 \sum_{k=1}^{\infty} \frac{(-1)^k}{\alpha^2 - k^2}, \quad \alpha\pi \operatorname{ctg} \alpha\pi = 1 + 2\alpha^2 \sum_{k=1}^{\infty} \frac{1}{\alpha^2 - k^2} \quad (\alpha \neq 0, \pm 1, \ldots).$$

9. If $\varphi_x(t)$ increases monotonically to $+\infty$ as $t \to +0$, $0 < t \leqslant t_0$, $\mathfrak{S}[f]$ diverges to $+\infty$ at the point x.

[Let $\varphi(t)/t = \chi(t)$. Then

$$\int_0^{\pi/n} \chi(t) \sin nt\,dt + o(1) \geqslant \pi s_n^*(x) \geqslant \int_0^{\pi/n} [\chi(t) - \chi(t + \frac{\pi}{n})] \sin nt\,dt + o(1) \geqslant$$

$$\geqslant \tfrac{1}{2}\int_0^{\pi/n} \chi(t) \sin nt\,dt + o(1)].$$

10. If (i) $\varphi_x(t) \to 0$ with t, (ii) $t\varphi'(t)$ is absolutely continuous except at $t=0$, (iii) $t\varphi'(t) > -A$, $A > 0$, for small $t > 0$, then $\mathfrak{S}[f]$ converges at x. Tonelli [1]; Hardy and Littlewood [3].

[Apply Young's test].

11. $\overline{\mathfrak{S}}[f]$ is convergent at the point x, provided that (1) the integral 2.61(1) exists and (2) the total variation of $t\psi(t)$ over $(0, h)$ is $O(h)$. See Prasad [1].

[The proof is analogous to that of Theorem 2.82 (ii) except at one point: to estimate $P = \int_0^{k/n} \psi(t)\,\overline{D}_n^*(t)\,dt$ we cannot use the fact that $\psi(t) \to 0$, but integrating by parts and applying condition (1) we find that $P \to 0$].

CHAPTER III.

Summability of Fourier series.

3.1. Toeplitz matrices. An infinite matrix

$$\mathfrak{A} = \left\{ \begin{array}{l} a_{00}, a_{01}, \dots a_{0n}, \dots \\ a_{10}, a_{11}, \dots a_{1n}, \dots \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{n0}, a_{n1}, \dots a_{nn}, \dots \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array} \right\}$$

is called a *Toeplitz* matrix, or *T*-matrix, if the following three conditions are satisfied (i) $\lim_i a_{ik} = 0$, $k = 0, 1, \dots$, (ii) $\lim_i A_i = 1$, (iii) $N_i \leqslant C$, $i = 0, 1, \dots$, where $A_i = a_{i0} + a_{i1} + \dots$, $N_i = |a_{i0}| + |a_{i1}| + \dots$ and C is independent of i. Given a sequence $\{s_n\}$, we 'transform' it by the matrix $\mathfrak{A}$, i. e. consider the sequence $\sigma_n = a_{n0} s_0 + a_{n1} s_1 + \dots$, provided that the series on the right converge. If $\sigma_n \to s$, we say that the sequence $\{s_n\}$, or the series with partial sums s_n, is summable $\mathfrak{A}$ to the value s. The expressions σ_n are called *T*-means.

If $\mathfrak{A}$ is a T-matrix and if $s_n \to s$, where s is finite, then $\sigma_n \to s$ [1]. In fact, if $s_k = s + \varepsilon_k$, $\varepsilon_k \to 0$, then $\sigma_n = \sigma'_n + \sigma''_n$, where $\sigma'_n = A_n s \to s$ (by (ii)). Given any $\varepsilon > 0$, suppose that $|\varepsilon_k| < \varepsilon/2C$ for $k > k_0$. Since $|\sigma''_n| \leqslant (|a_{n0}|\,|\varepsilon_0| + \dots + |a_{nk_0}|\,|\varepsilon_{k_0}|) + (|a_{n\,k_0+1}|\,|\varepsilon_{k_0+1}| + \dots)$, where the second sum on the right is less than $C \cdot \varepsilon/2C = \varepsilon/2$, and the first sum tends to 0 (by (i)), it follows that $|\sigma''_n| < \varepsilon/2 + \varepsilon/2 = \varepsilon$ for n large, i. e. $\sigma''_n \to 0$, $\sigma_n \to s$.

It is useful to note that, if $s = 0$, condition (ii) is not necessary in the proof. If s_n depends on a parameter and if $s_n \to s$ uniformly, then $\sigma_n \to s$ also uniformly.

---

[1]) Toeplitz [1].

3.101. Condition (iii) is a consequence of (ii) if $\mathfrak{A}$ is *positive* i. e. if all a_{ik} are non-negative. For such matrices we can prove the following more general result:

$$\underline{\lim}\, s_n \leqslant \underline{\lim}\, \sigma_n \leqslant \overline{\lim}\, \sigma_n \leqslant \overline{\lim}\, s_n.$$

To prove e. g. the first inequality we may plainly suppose that $\underline{\lim}\, s_n = \underline{s} > -\infty$. Let α be any number $< \underline{s}$. Then $s_k > \alpha$ for $k > k_0$, and so, by (i), we have the inequality $\sigma_n \geqslant o(1) + (a_{n\,k_0+1} + \dots)\,\alpha = o(1) + \alpha\,[A_n + o(1)]$, and therefore $\underline{\lim}\, \sigma_n \geqslant \alpha$, $\underline{\lim}\, \sigma_n \geqslant \underline{s}$. In particular if $s_n \to \infty$, then $\sigma_n \to \infty$.

If $\mathfrak{A}$ is not positive the result is not necessarily true. A moment's consideration shows that, if $\underline{\lim}\, s_n = \underline{s}$, $\overline{\lim}\, s_n = \overline{s}$, $\lim N_i = C$, then $\underline{\lim}\, \sigma_n$ and $\overline{\lim}\, \sigma_n$ are both contained in the interval $[\frac{1}{2}(\underline{s} + \overline{s}) - C \cdot \frac{1}{2}(\overline{s} - \underline{s}), \frac{1}{2}(\underline{s} + \overline{s}) + C \cdot \frac{1}{2}(\overline{s} - \underline{s})]$. In fact, we may put $s_n = s'_n + s''_n$, where $s'_n = \frac{1}{2}(\underline{s} + \overline{s})$, $\overline{\lim}\, |s''_n| = \frac{1}{2}(\overline{s} - \underline{s})$. Then $\sigma_n = \sigma'_n + \sigma''_n$, where $\sigma'_n \to \frac{1}{2}(\underline{s} + \overline{s})$ and $\overline{\lim}\, \sigma''_n \leqslant C \cdot \frac{1}{2}(\overline{s} - \underline{s})$.

3.102. Let $\{p_n\}$, $\{q_n\}$ be two sequences of numbers, and let $P_n = p_0 + \dots + p_n$, $Q_n = q_0 + \dots + q_n$, $q_n > 0$, $Q_n \to \infty$. *If* $s_n = p_n/q_n \to s$, *then* $\sigma_n = P_n/Q_n \to s$. In fact, $\sigma_n = (q_0 s_0 + q_1 s_1 + \dots + q_n s_n)/Q_n$, so that we have here a positive T-matrix. In particular, if $q_n = 1$ for $n = 0, 1, \dots$, we obtain the classical result of Cauchy: *if* $s_n \to s$, *then* $(s_0 + s_1 + \dots + s_n)/(n+1) \to s$.

3.11. Cesàro means. Given a sequence $\{s_n\}$ we put, for $n = 0, 1, \dots$, $s_n^0 = s_n$, $s_n^1 = s_0^0 + s_1^0 + \dots + s_n^0, \dots, s_n^k = s_0^{k-1} + s_1^{k-1} + \dots + s_n^{k-1}, \dots$ Similarly, let $A_n^0 = 1$ $(n = 0, 1, \dots)$, $A_n^1 = A_0^0 + A_1^0 + \dots + A_n^0, \dots, A_n^k = A_0^{k-1} + A_1^{k-1} + \dots + A_n^{k-1}, \dots$ We shall say that the sequence $\{s_n\}$ is summable by the k-th Cesàro mean, or summable (C, k), $k = 0, 1, \dots$, to limit s, if $s_n^k / A_n^k \to s$ as $n \to \infty$. (It follows from § 3.102 that summability (C, k) of a sequence involves summability $(C, k+1)$ to the same limit [1]). To find the numerical values of A_n^k it is con-

[1]) Les us define, for every $k = 0, 1, \dots$, the sequence $h_n^k = (h_0^{k-1} + \dots + h_n^{k-1})/(n+1)$, $n = 0, 1, \dots$, $h_n^0 = s_n$. $\{s_n\}$ is said to be summable by the k-th Hölder mean, or summable (H, k), if $h_n^k \to s$ as $n \to \infty$. The methods (C, k) and (H, k) are known to be equivalent. Although the latter is logically simpler, it is less useful in applications and its extension to the case of k non-integral much less easy. See Hausdorff [1].

venient to use the following proposition, which is easily proved by means of Abel's transformation: If $A_n = a_0 + a_1 + \dots + a_n$, then

$$\sum_{n=0}^{\infty} a_n x^n = (1-x)\sum_{n=0}^{\infty} A_n x^n,$$

provided that the series on the right is convergent. This permits us to restate our definition as follows:

A sequence $\{s_n\}$, or a series $u_0 + u_1 + \dots$ with partial sums s_n, is summable (C, α) to the value s if $\sigma_n^\alpha = s_n^\alpha / A_n^\alpha \to s$, s_n^α and A_n^α being given by the relations

$$(1) \quad \sum_{n=0}^{\infty} A_n^\alpha x^n = (1-x)^{-\alpha-1}, \quad \sum_{n=0}^{\infty} s_n^\alpha x^n = \frac{\sum_{n=0}^{\infty} s_n x^n}{(1-x)^\alpha} = \frac{\sum_{n=0}^{\infty} u_n x^n}{(1-x)^{\alpha+1}}.$$

In this definition $\alpha\,(\neq -1, -2, \dots)$ is no longer a positive integer. However it will appear soon that only the case $\alpha > -1$ is interesting. The following relations are consequences of (1):

$$(2) \quad A_n^\alpha = \binom{n+\alpha}{n} = \frac{(\alpha+1)\dots(\alpha+n)}{n!} \simeq \frac{n^\alpha}{\Gamma(\alpha+1)}, \quad \alpha \neq -1, -2, \dots$$

$$(3) \quad A_n^{\alpha+\beta+1} = \sum_{k=0}^{n} A_k^\alpha A_{n-k}^\beta, \qquad (4) \quad s_n^{\alpha+\beta+1} = \sum_{k=0}^{n} A_{n-k}^\beta s_k^\alpha,$$

$$(5) \quad s_n^\alpha = \sum_{k=0}^{n} A_{n-k}^{\alpha-1} s_k = \sum_{k=0}^{n} A_{n-k}^\alpha u_k, \quad (6) \quad A_n^\alpha = \sum_{k=0}^{n} A_k^{\alpha-1}, \; A_n^\alpha - A_{n-1}^\alpha = A_n^{\alpha-1},$$

$$(7) \quad s_n^\alpha = \sum_{k=0}^{n} s_k^{\alpha-1}, \; s_n^\alpha - s_{n-1}^\alpha = s_n^{\alpha-1}, \quad (8) \quad \sum_{k=0}^{\infty} |A_k^\alpha| < \infty, \; \alpha < -1, {}^{1)}$$

(9) A_n^α is positive for $\alpha > -1$, increasing for $\alpha > 0$, and decreasing for $0 > \alpha > -1$. If $\alpha < -1$, A_n^α is of constant sign for n sufficiently large.

3.12. The Gamma-function. In 3.11(2) Γ is the Euler Gamma-function. Except in Chapter IX, the reader is not expected to be acquainted with the theory of this function, and may take the relation 3.11(2) just as a definition (Gauss's definition) of Γ. It remains, then, only to show that $\lim A_n^\alpha / n^\alpha$ exists and is different from 0. For this purpose we write

---

1) See (2).

$$\log A_n^\alpha = \sum_{k=1}^{n} \log\left(1+\frac{\alpha}{k}\right) = \sum_{k=0}^{n}\{\frac{\alpha}{k} + O(k^{-2})\} = \alpha(\log n + C + \varepsilon_n) + (C' + \eta_n),$$

where C is Euler's constant (§ 1.74), $\varepsilon_n, \eta_n \to 0$, and C' is the sum of all the terms $O(k^{-2})$. This completes the proof.

3.13. *If* $\sigma_n^\alpha \to s$, $\alpha > -1$, $h > 0$, *then* $\sigma_n^{\alpha+h} \to s$. We obtain from 3.11(4) that $\sigma_n^{\alpha+h} = \left(\sum_{k=0}^{n} A_{n-k}^{h-1} A_k^\alpha \sigma_k^\alpha\right)/A_n^{\alpha+h}$. This is a positive T-matrix, and so the result follows. Also, more generally, the limits of indetermination of $\sigma_n^{\alpha+h}$ are contained between those of σ_n^α.

If $u_0 + u_1 + \ldots$ *is summable* (C, α), *and if* $\alpha > -1$, *then* $u_n = o(n^\alpha)$. We have $u_n/A_n^\alpha = \left(\sum_{k=0}^{n} A_{n-k}^{-\alpha-2} A_k^\alpha \sigma_k^\alpha\right)/A_n^\alpha$ (§ 3.11(4), $\beta = -\alpha - 2$). Suppose, as we may, that $\sigma_k^\alpha \to 0$. We need only show that conditions (i) and (iii) of Toeplitz are satisfied (§ 3.1). The former is obviously satisfied. As regards the latter, let us suppose first $\alpha \geqslant 0$. Then, A_k^α being non-decreasing, we have $N_n \leqslant \sum_{k=0}^{n} |A_k^{-\alpha-2}| = O(1)$. If $-1 < \alpha < 0$, we obtain from 3.11(3) that $N_n = 2$, since $A_k^{-\alpha-2}$ is negative for $k > 0$.

It is often useful to consider the difference

(1) $$s_n - \sigma_n^1 = (u_1 + 2u_2 + \ldots + nu_n)/(n+1).$$

If it tends to 0, in particular if $u_n = o(1/n)$, the $(C, 1)$ summability of $u_0 + u_1 + \ldots$ involves the convergence of this series.

3.14. Abel's method of summation. The series $u_0+u_1+\ldots$ is said to be summable by Abel's method (some say Poisson's), or summable A, to sum s, if $u_0 + u_1 x + u_2 x^2 + \ldots$ is convergent for $|x| < 1$, and

(1) $$\lim_{x\to1} \sum_{k=0}^{\infty} u_k x^k = \lim_{x\to1} (1-x) \sum_{k=0}^{\infty} s_k x^k = s,$$

where x tends to 1 along the real axis.

If $u_0 + u_1 + \ldots$ *is summable* (C, α), $\alpha > -1$, *to* s, *then* (1) *holds as* $x \to 1$ *along any path* L *lying between two chords of the unit circle which pass through* $x = 1$. Such paths L will be spoken of as not touching the circle. They are characterized by inequalities $|1-x|/(1-|x|) < \text{const.}, x \in L$.

To avoid the difficulty that the variable x in (1) changes continuously, we consider an arbitrary sequence of points x_n lying on L and tending to 1. Since

$$\sum_{k=0}^{\infty} u_k x_n^k = (1-x_n)^{\alpha+1} \sum_{k=0}^{\infty} s_k^\alpha x_n^k = (1-x_n)^{\alpha+1} \sum_{k=0}^{\infty} \sigma_k^\alpha A_k^\alpha x_n^k,$$

we need only show that the matrix $\mathfrak{A}$ with $a_{nk} = A_k^\alpha (1-x_n)^{\alpha+1} x_n^k$ is a T-matrix. If $x_n \to 1$ along the real axis, the matrix is positive, so that the limits of indetermination by the method A are contained between those by the method (C, α).

3.2. As we shall see in Chapter VIII, there exist continuous functions with Fourier series divergent at some points. It is therefore natural to consider the summability of Fourier series. Although some older results, e. g. those of Poisson, in the theory of trigonometrical series can now be recognized as applications of methods of summability, the first deliberate step in this direction was made by Fejér (1902). The results proved in this chapter, together with the examples of Chapter VIII, show that, if we do not restrict ourselves to functions with rather special differential properties, it is rather the summability than the ordinary convergence which is important in the theory of representation of functions by means of their Fourier series.

3.201. Let $s_n(x)$ be the n-th partial sum of $\mathfrak{S}[f]$:

(1) $$f(x) \sim \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

and let $\sigma_n(x) = \sigma_n(x; f)$ be the first arithmetic means of $\{s_n\}$.

Using the formulae 2.3(2), we see that

(2) $$\sigma_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) K_n(t)\, dt,$$

$$\sigma_n(x) - f(x) = \frac{1}{\pi} \int_0^{\pi} \varphi(t) K_n(t)\, dt,$$

where, as usual, $\varphi(t) = \varphi_x(t) = f(x+t) + f(x-t) - 2f(x)$, and $K_n = (D_0 + D_1 + \dots + D_n)/(n+1)$. Multiplying the numerator and the denominator of $D_k(t)$ by $2 \sin \frac{1}{2} t$, and replacing the products of sines by differences of cosines, we find that

$$(3)\qquad (n+1)K_n(t)=\sum_{k=0}^{n}\frac{\sin(k+\frac{1}{2})t}{2\sin\frac{1}{2}t}=\frac{1}{2}\left(\frac{\sin(n+1)\frac{1}{2}t}{\sin\frac{1}{2}t}\right)^2.$$

It is customary, in general, in the theory of Fourier series to call the Toeplitz means of the series $\frac{1}{2}+\cos t+\cos 2t+\ldots$ *kernels*. The expression $K_n(t)$ is called 'Fejér's kernel' and has the following properties:

(i) $K_n(t)\geqslant 0$, (ii) $\frac{1}{\pi}\int_{-\pi}^{\pi}K_n(t)\,dt=1$, (iii) $M_n(\delta)\to 0$ as $n\to\infty$, for every $\delta>0$, where $M_n(\delta)=\text{Max}\,|K_n(t)|=\text{Max}\,K_n(t)$ for $\delta\leqslant t\leqslant\pi$, $n=0,1,\ldots$

Condition (ii) follows from the analogous property of D_n, and (iii) from the inequality $M_n(\delta)\leqslant 1/2\,(n+1)\sin^2\frac{1}{2}\delta$. Kernels with such properties are called *positive* kernels. Kernels satisfying, besides (ii), (iii), the condition (i') $\int_{-\pi}^{\pi}|K_n(t)|\,dt\leqslant C$ will be called 'quasi-positive'. Condition (i') follows from (ii) if (i) is satisfied.

3.21. Fejér's theorem [1]. *If the limits $f(x\pm 0)$ exist, $\mathfrak{S}[f]$ is summable $(C,1)$ at the point x to the value $\frac{1}{2}[f(x+0)+f(x-0)]$. In particular, if f is continuous at x, $\mathfrak{S}[f]$ is summable there to the value $f(x)$. If f is continuous at every point of an interval $I=(a,b)$* [2], *$\mathfrak{S}[f]$ is uniformly summable in I.*

The proof will be based only on the properties (i), (ii), (iii) of K_n. We may assume that $2f(x)=f(x+0)+f(x-0)$, so that $|\varphi_x(t)|<\varepsilon$ for $0\leqslant t\leqslant\delta=\delta(\varepsilon)$. From 3.201(2) we see that $|\sigma_n(x)-f(x)|$ does not exceed

$$(1)\qquad \frac{1}{\pi}\int_0^{\pi}|\varphi(t)|\,K_n(t)\,dt=\frac{1}{\pi}\left(\int_0^{\delta}+\int_{\delta}^{\pi}\right)\leqslant\frac{\varepsilon}{\pi}\int_0^{\pi}K_n\,dt+\frac{M_n(\delta)}{\pi}\int_0^{\pi}|\varphi|\,dt.$$

Let us denote the last two terms by P, Q. We have $P=\varepsilon/2$ (cond. (ii)), $Q\to 0$ (cond. (iii)), so that $P+Q<\varepsilon$ for $n>n_0=n_0(\varepsilon)$, and, ε being arbitrary, the first part of the theorem follows.

---

[1] Fejér [1].

[2] We mean by this that f is continuous also at the points a, b.

If f is continuous at every point of I, we can find a δ such that $|\varphi_x(t)| < \varepsilon$ for $0 \leqslant t \leqslant \delta$, $x \in I$, and so (1) holds for any $x \in I$. The integral in Q does not exceed

$$\int_0^{\pi} |(f(x+t)| + |f(x-t)| + 2|f(x)|)\, dt = \int_{-\pi}^{\pi} |f(t)|\, dt + 2\pi\, |f(x)|.$$

Hence $Q \to 0$ uniformly in I, so that $P+Q < \varepsilon$ for $n > n_0$, $x \in I$.

If, in particular, (a, b) coincides with $(0, 2\pi)$, $\sigma_n(x)$ converges uniformly to $f(x)$.

3.211. The theorem would be true even if K_n were only quasi-positive. In fact, K_n in 3.21(1) should then be replaced by $|K_n|$. We should have $P = C\varepsilon/2$, $Q \to 0$, i. e. $P+Q < C\varepsilon$ for $n > n_0$.

3.22. *If* $m \leqslant f(x) \leqslant M$ *in* $(0, 2\pi)$, *then* $m \leqslant \sigma_n(x) \leqslant M$ [1], i. e. the Fejér means are contained in the same range as the function f. (In particular $\sigma_n \geqslant 0$ if $f \geqslant 0$). This follows from the first formula 3.201(2) if we replace $f(x+t)$ first by m, and then by M, and take into account conditions (i), (ii).

If $m \leqslant f(x) \leqslant M$ *for* $x \in I = (a, b)$, *then, for every* $\delta > 0$, *there exists an integer* $n_0 = n_0(\delta)$ *such that*

$$(1) \quad m - \delta \leqslant \sigma_n(x) \leqslant M + \delta, \quad \textit{for } x \in I_\delta = (a+\delta, b-\delta),\ n > n_0.$$

Break up the first integral 3.201(2) into three, extended over $(-\pi, -\delta)$, $(-\delta, \delta)$, (δ, π), and denote them by σ_n', σ_n'', σ_n'''. If $x \in I_\delta$, $|t| < \delta$, then $x + t \in I$, and σ_n'' is contained between m and M, multiplied by the integral of $K_n(t)/\pi$ over $(-\delta, \delta)$. In virtue of conditions (ii) and (iii) this last integral tends to 1. Since $|\sigma_n'|$ and $|\sigma_n'''|$ do not exceed $M_n(\delta)/\pi$ multiplied by the integral of $|f(t)|$ over $(-\pi, \pi)$, and so tend to 0, a moment's consideration shows that (1) is valid.

From (1) we obtain in particular that $m \leqslant \underline{\lim}\, \sigma_n(x) \leqslant \overline{\lim}\, \sigma_n(x) \leqslant M$, for every $a < x < b$.

Given a function $f(x)$ let $M(a, b)$ and $m(a, b)$ denote the upper and lower bound respectively of f in (a, b). For every x let $M(x) = \lim M(x-h, x+h)$, $m(x) = \lim m(x-h, x+h)$ as $h \to 0$. The numbers $M(x)$, $m(x)$ are called the maximum and minimum respectively of f at the point x. From the last remark it follows

[1] More precisely $m < \sigma_n(x) < M$, unless $f \equiv$ const.

that, for every x, $m(x) \leqslant \underline{\lim}\, \sigma_n(x) \leqslant \overline{\lim}\, \sigma_n(x) \leqslant M(x)$. If in particular $m(x) = M(x) = \infty$, then $\sigma_n(x) \to \infty$.

3.23. Corollaries of Fejér's theorem. (i) *If $\mathfrak{S}[f]$ converges at a point where f is continuous, or has a simple discontinuity, then it converges to $\frac{1}{2}[f(x+0)+f(x-0)]$.* In fact, if a series converges to s, it is summable $(C, 1)$ to the same value.

More generally, if x is a point of continuity of f, the interval of oscillation of the partial sums $s_n(x)$ contains $f(x)$.

(ii) *If f is of bounded variation, the partial sums of $\mathfrak{S}[f]$ are uniformly bounded.* Since the $\sigma_n(x; f)$ are uniformly bounded, it is sufficient to observe that the Fourier coefficients of f are $O(1/n)$ (§ 2.213) and to use the formula 3.13(1).

(iii) *If f is continuous and of period 2π, there exists, for every $\varepsilon > 0$, a trigonometrical polynomial $T(x)$ such that $|f(x) - T(x)| < \varepsilon$ everywhere.* We may take for $T(x)$ the expressions $\sigma_n(x; f)$ with n sufficiently large.

(iv) *The trigonometrical system is complete* (§ 1.5). If all the Fourier coefficients of a continuous function f vanish, $f(x)$, as the limit of Fejér's means, vanishes identically. For the case of discontinuous f see the argument in § 1.5.

(v) Hardy observed that *Dirichlet's Theorem* (§ 2.6) *can be deduced from Fejér's* by means of the following theorem from the general theory of series: *If $u_0 + u_1 + \dots$ is summable $(C, 1)$ to a sum s and $|u_n| \leqslant A/n$, $n = 1, 2, \dots$, where A is a constant, the series is convergent*[1]).

Without loss of generality we may assume that $s = 0$, $A = 1$. Let p, $p < n$, be a function of n tending to $+\infty$ which we shall define presently. Since $\sigma_p^1 \to 0$, the relation

$$\sigma_n^1 = \frac{s_0 + \dots + s_p}{n+1} + \frac{s_{p+1} + \dots + s_n}{n+1} \to 0 \text{ involves } \frac{s_{p+1} + \dots + s_n}{n+1} \to 0.$$

If $k < n$, then $|s_n - s_k| \leqslant |u_{k+1} + \dots + u_n| < 1/(k+1) + \dots + 1/n < (n-k)/k$ and so the last relation may be written in the form

$$(1) \qquad \frac{n-p}{n+1} s_n + \theta \cdot \frac{(n-p)(n-p+1)}{2p(n+1)} \to 0,$$

[1]) Hardy [5].

where $\theta = \theta(n, p)$ does not exceed 1 in absolute value. Put now $n - p = [\varepsilon n]$, i. e. $p = n - [\varepsilon n]$, where $0 < \varepsilon < 1/2$ is arbitrary but fixed. Dividing both sides of (1) by $(n-p)/(n+1)$, we see that $\overline{\lim}\, |s_n| \leqslant \varepsilon/2(1-\varepsilon) < \varepsilon$, that is $s_n \to 0$.

Although the above argument is, on the whole, not simpler than the direct proof of Dirichlet's theorem, it is interesting as an application of the theory of summability to the convergence of Fourier series.

3.3. Summability (C, r) of Fourier series. *Fejér's theorem remains true if we replace summability $(C, 1)$ by (C, r), $r > 0$* [1]. Denoting the (C, r) means of $\mathfrak{S}[f]$ by $\sigma_n^r(x)$, we find from 3.11(5), 2.3(2) the formulae

$$\sigma_n^r(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\, K_n^r(t)\, dt, \quad \sigma_n^r(x) - f(x) = \frac{1}{\pi}\int_0^{\pi} \varphi_x(t)\, K_n^r(t)\, dt,$$

(1)
$$K_n^r(t) = \sum_{k=0}^{n} A_{n-k}^{r-1} D_k(t) / A_n^r,$$

and it is sufficient to show that the kernel K_n^r is quasi-positive. We may suppose that $0 < r \leqslant 1$. Condition (ii) of § 3.201 is obviously satisfied. Conditions (i) and (iii) follow from the inequalities

$$(2) \qquad |K_n^r(t)| \leqslant 2n, \quad |K_n^r(t)| \leqslant C n^{-r} t^{-r-1} \quad \text{for } 1/n \leqslant t \leqslant \pi,$$

which we will now prove; C is a constant independent of n. From the formula defining K_n^r we obtain

$$K_n^r(t) = \frac{1}{2A_n^r \sin\frac{1}{2}t}\, \mathfrak{I} \sum_{k=0}^{n} A_{n-k}^{r-1} e^{i(k+\frac{1}{2})t} = \mathfrak{I}\left[\frac{e^{i(n+\frac{1}{2})t}}{2A_n^r \sin\frac{1}{2}t} \sum_{k=0}^{n} A_k^{r-1} e^{-ikt}\right] =$$

(3)
$$= \mathfrak{I}\left\{\frac{e^{i(n+\frac{1}{2})t}}{2A_n^r \sin\frac{1}{2}t}\left[(1-e^{-it})^{-r} - \sum_{k=n+1}^{\infty} A_k^{r-1} e^{-ikt}\right]\right\}.$$

Since A_n^{r-1} decreases steadily to 0, the last series converges for $t \neq 0$ and its sum does not exceed $4A_{n+1}^{r-1}/|1-e^{-it}|$ in absolute value (§ 1.23). And since $|\mathfrak{I}(z)| \leqslant |z|$, we have that, for $0 < t \leqslant \pi$, $|K_n^r(t)|$ does not exceed

$$\{(2\sin\tfrac{1}{2}t)^{-r-1} + 4A_{n+1}^{r-1}(2\sin\tfrac{1}{2}t)^{-2}\}/A_n^r \leqslant \tfrac{1}{2}\, C\,(n^{-r} t^{-r-1} + n^{-1} t^{-2}).$$

---

[1] M. Riesz [1], [2]; Chapman [1].

Taking into account that $nt^2 = n^r t^{r+1} (nt)^{1-r} \geqslant n^r t^{r+1}$ for $nt \geqslant 1$, we obtain the second inequality (2). To prove the first, we note that $|D_k(t)| \leqslant \frac{1}{2} + 1 + \dots + 1 = k + \frac{1}{2} \leqslant n + 1$ for $0 \leqslant k \leqslant n$, and so, applying 3.11(6), we obtain from (1) that $|K_n^r(t)| < n+1 \leqslant 2n$ $(n > 0)$.

It is of some interest to note that for $r=1$ the formulae (2) are consequences of 3.201(3).

3.31. *$\mathfrak{S}[f]$ is summable (C, r), $r > 0$, to the value $f(x)$ at every point x where $\Phi_x(t) = o(t)$* [1] *and so, in particular, almost everywhere* (§ 2.703). This theorem is a simple consequence of 3.3(2). In fact

$$\pi |\sigma_n^r(x) - f(x)| \leqslant \left(\int_0^{1/n} + \int_{1/n}^{\pi}\right) |\varphi_x(t)|\,|K_n^r(t)|\,dt = P + Q.$$

From the first inequality in 3.3(2) we see that $P \leqslant 2n\,\Phi_x(1/n) \to 0$.

Integrating by parts we find that $Q \leqslant Cn^{-r}[\Phi_x(t)\,t^{-r-1}]_{1/n}^{\pi} +$

$$+ C(1+r)\,n^{-r}\int_{1/n}^{\pi} \Phi_x(t)\,t^{-r-2}\,dt = o(1) + C(1+r)\,n^{-r}\int_{1/n}^{\pi} o(t^{-1-r})\,dt = o(1)$$

(§ 1.71).

3.32. Summability (C, r) of conjugate series. Let $\bar{\sigma}_n^r$ denote the (C, r) means of $\bar{\mathfrak{S}}[f]$.

For almost every x the difference

(1) $$\bar{\sigma}_n^r(x) - \left(-\frac{1}{\pi}\int_{1/n}^{\pi} \psi_x(t)\,\tfrac{1}{2}\operatorname{ctg}\tfrac{1}{2}t\,dt\right), \quad 0 < r \leqslant 1,$$

where $\psi_x(t) = f(x+t) - f(x-t)$, tends to 0 as $n \to \infty$. This is in particular true for every x where $\Psi_x(t) = o(t)$ (§ 2.703)[2]. The proof is, roughly, the same as in Theorem 3.31. We have

(2) $$\bar{\sigma}_n^r(x) = -\frac{1}{\pi}\int_0^{\pi} \psi_x(t)\,\bar{K}_n^r(t)\,dt = -\frac{1}{\pi}\left(\int_0^{1/n} + \int_{1/n}^{\pi}\right) = A + B,$$

$$\bar{K}_n^r(t) = \frac{1}{A_n^r}\sum_{k=0}^{n} A_{n-k}^{r-1}\,\bar{D}_k(t) = \tfrac{1}{2}\operatorname{ctg}\tfrac{1}{2}t - \frac{1}{A_n^r}\sum_{k=0}^{n} A_{n-k}^{r-1}\frac{\cos(k+\frac{1}{2})t}{2\sin\frac{1}{2}t}.$$

---

[1] See Lebesgue [3] for $r=1$, Hardy [2] for the general case.

[2] See Privaloff [2], Plessner [2] for $r=1$, Hardy and Littlewood [4], Zygmund [2] for the general case.

Exactly in the same way as in § 3.3 we show that $|K_n^r(t)| < 2n$, so that $A \to 0$, and the difference (1) is equal to

$$(3) \qquad \frac{1}{\pi}\int_{1/n}^{\pi} \psi_x(t)\, H_n^r(t)\, dt + o(1),$$

where $H_n^r(t)$ denotes the last sum in (2). For $H_n^r(t)$ we obtain the expression 3.3(3) with $\mathfrak{I}$ replaced by $\mathfrak{R}$. It follows that $H_n^r(t)$ satisfies the second inequality in 3.3(2), which, as we have shown in § 3.31, is sufficient to prove that (3) tends to 0.

3.321. The result of the preceding section shows that, for almost every x, the summability (C, r), $r > 0$, of $\bar{\mathfrak{S}}[f]$ is equivalent to the existence of the integral

$$(1) \qquad -\frac{1}{\pi}\int_0^{\pi} \psi_x(t)\, \tfrac{1}{2}\operatorname{ctg} \tfrac{1}{2} t\, dt = \lim_{h\to 0}\left(-\frac{1}{\pi}\int_h^{\pi} \psi_x(t)\, \tfrac{1}{2}\operatorname{ctg}\tfrac{1}{2} t\, dt\right)$$ [1].

The problem of the existence of this integral is very delicate. We shall show in Chapter VII that it exists almost everywhere, for every integrable f. Taking this result here for granted, we obtain that $\bar{\mathfrak{S}}[f]$ is summable (C, r), $r > 0$, almost everywhere, to the value $\bar{f}(x)$ given in (1).

3.4. Abel's summability. In connection with 3.201(1) we put, for $0 \leqslant r < 1$,

$$f(r, x) = \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)\, r^n,$$

$$\bar{f}(r, x) = \sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx)\, r^n.$$

Taking into account 1.12(1) and 1.12(2), we easily find that

$$f(r, x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\, P_r(t)\, dt, \quad f(r, x) - f(x) = \frac{1}{\pi}\int_0^{\pi} \varphi_x(t)\, P_r(t)\, dt,$$

$$(1) \qquad \bar{f}(r, x) = -\frac{1}{\pi}\int_0^{\pi} \psi_x(t)\, Q_r(t)\, dt.$$

[1]) If $\bar{f}(x, h)$ denotes the second integral in (1), and if $1/(n+1) < h \leqslant 1/n$, then $\pi\,|\bar{f}(x, h) - \bar{f}(x, 1/n)| \leqslant (n+1)\, \Psi_x(1/n) \to 0$, as $n \to \infty$.

The functions

$$(2)\qquad P_r(t)=\tfrac{1}{2}(1-r^2)/\Delta_r(t),\ Q_r(t)=r\sin t/\Delta_r(t),$$
$$\text{where } \Delta_r(t)=1-2r\cos t+r^2,\ 0\leqslant r<1,$$

are called, for historical reasons, *Poisson's kernel* and *Poisson's conjugate kernel*. The expression on the right in the first formula (1) is called *Poisson's integral*. It is not difficult to see that $P_r(t)$ is a positive kernel, i. e. satisfies the conditions (i), (ii), (iii) of § 3.201. That r, which now plays the rôle of the index n, is a continuous variable, is irrelevant. Condition (i) follows from the inequality $\Delta_r(t)>0$. Condition (ii) may be obtained integrating both sides of 1.12(1) over the range $(-\pi,\pi)$. Since $\Delta_r(t)=$ $=(1-r)^2+4r\sin^2\frac{1}{2}t$, we see that $M_r(\delta)=\text{Max}\,P_r(t)$ for $0<\delta\leqslant t\leqslant\pi$ is $\leqslant(1-r^2)/8r\sin^2\frac{1}{2}\delta\to 0$ as $r\to 1$, so that condition (iii) is also fulfilled. Hence

Theorem 3.21 *remains true if we replace summability* $(C,1)$ *by summability A.* The reader has, no doubt, noticed, that this theorem is a consequence of Fejér's theorem and of Theorem 3.14, but a direct study of Poisson's kernel is interesting in itself.

3.41. The functions $f(r,x)$, $\bar{f}(r,x)$, as the real and imaginary parts of a function analytic inside the unit circle (§ 1.12), are harmonic, that is, when treated as functions of rectangular coordinates ξ, η, they satisfy Laplace's equation $\partial^2u/\partial\xi^2+\partial^2u/\partial\eta^2=0$. Let us denote the polar coordinates of points in the unit circle by r, x $(0\leqslant r<1,\ 0\leqslant x<2\pi)$, and let $f(x)$ be a continuous and periodic function of x. The function $f(r,x)$ defined by Poisson's integral tends uniformly to $f(x)$ as $r\to 1$. In other words, Poisson's integral gives a solution (or rather, as it is well-known, *the* solution) for the case of the unit circle of the following very famous problem ('Dirichlet's problem'): Given (1) a plane region G, whose boundary is a simple closed curve L, (2) a function $f(p)$, defined and continuous at the points $p\in L$, to find a function $F(p)$, harmonic in G, continuous in $G+L$, and coinciding with $f(p)$ on L. However, in this special case of the unit circle, Poisson's integral gives a solution of a more general Dirichlet's problem, viz. when the limit function is an arbitrary integrable function (§ 3.442).

3.42. *If* $m\leqslant f(x)\leqslant M$, *then* $m\leqslant f(r,x)\leqslant M$. *If* $m\leqslant f(x)\leqslant M$ *for* $x\in I=(a,b)$, *then, for every* $\delta>0$, *there exists a number* r_0

such that $m-\delta \leqslant f(r, x) \leqslant M+\delta$ *for* $x \in (a+\delta, b-\delta)$, $r_0 \leqslant r < 1$. The proof is essentially the same as in § 3.22.

If $M(x_0)$ *and* $m(x_0)$ *are the maximum and minimum of* f *at a point* x_0 (§ 3.22), *and if* L *is an arbitrary path leading from inside the unit circle to the point* $(1, x_0)$, *the limits of indetermination of* $f(r, x)$, *as the point* (r, x) *approaches* $(1, x_0)$ *along* L, *are contained between* $m(x_0)$ *and* $M(x_0)$. In fact, given an $\varepsilon > 0$, there exists an h such that $m(x_0)-\varepsilon \leqslant f(x) \leqslant M(x_0)+\varepsilon$ for $|x-x_0| \leqslant h$. Supposing, as we may, that $h < \varepsilon$, let us apply the preceding theorem with $(a, b) = (x_0 - h, x_0 + h)$, $\delta = h/2$. Then, if (r, x) belongs to the curvilinear quadrangle (Q) $r_0 < r < 1$, $|x - x_0| < h/2$, $f(r, x)$ is contained between $m(x_0) - \varepsilon - h/2$ and $M(x_0) + \varepsilon + h/2$, and à fortiori between $m(x_0) - 3\varepsilon/2$ and $M(x_0) + 3\varepsilon/2$. Since, from some point onwards, L lies entirely in Q, and ε is arbitrary, the theorem follows[1]. In particular, if f is continuous at x_0, $\lim f(r, x)$ along L exists and is equal to $f(x_0)$.

3.43. Let x_0 be a point of simple discontinuity for f. To determine the behaviour of $f(r, x)$ in the neighbourhood of $(1, x_0)$, suppose that $x_0 = 0$, $2f(0) = f(+0) + f(-0)$, $d = f(+0) - f(-0) \neq 0$. Let $\delta(x)$ denote the periodic function equal to $(\pi - x)/2$ for $0 < x < 2\pi$. The difference $g(x) = f(x) - \delta(x)\, d/\pi$ is continuous at $x = 0$, and $g(0) = f(0)$. If $g(r, x)$ and $\delta(r, x)$ are Poisson's integrals for g and δ, then $f(r, x) = g(r, x) + \delta(r, x)\, d/\pi$. Let α be the angle at which a path L meets the real axis at the point $(1, 0)$, that is $\alpha = \lim \beta$, where β is the angle of the vector $(1, 0)\,(r, x)$ with the real axis. Since $g(r, x) \to g(0) = f(0)$, and $\delta(r, x) = \operatorname{arctg}\{r \sin x/(1 - r\cos x)\}$ (§ 1.12(3)), we see that $f(r, x)$ tends to $f(0) + \alpha\, d/\pi$, i. e. *the limit is a linear function of the angle at which* L *meets the radius at the point* $(1, x_0)$. It is plain that if $\alpha = \lim \beta$ does not exist, $f(r, x)$ oscillates finitely as $(r, x) \to (1, x_0)$ along L.

3.44. Fatou's theorems[2]. Let $F(x)$ be a function with Fourier coefficients A_n, B_n. *If* $[F(x+t) - F(x-t)]/2t \to l$ *as* $t \to 0$, *where* l *is not necessarily finite, then* $\mathfrak{S}'[F]$ *is summable* A *at the point* x *to the value* l, *i. e.*

---

[1]) The corresponding result for Fejér's means is as follows: for every $\{h_n\} \to 0$, the limits of indetermination of $\{\sigma_n(x_0 + h_n)\}$ are contained between $m(x_0)$ and $M(x_0)$.

[2]) Fatou [1]. See also Grosz [1].

(1) $$\sum_{n=1}^{\infty} n\,(B_n \cos nx - A_n \sin nx)\, r^n = \frac{\partial F(r,x)}{\partial x} \to l \text{ as } r \to 1.$$

More generally, if $l_1 \leqslant l_2$ are the limits of indetermination of the ratio $[F(x+t) - F(x-t)]/2t$, as $t \to 0$, the limits of indetermination of the expression in (1) are contained between l_1 and l_2 [1]). We have

(2) $$F(r,x) = \frac{1}{\pi}\int_{-\pi}^{\pi} F(t)\, P_r(t-x)\, dt,$$

$$\frac{\partial F(r,x)}{\partial x} = -\frac{1}{\pi}\int_{-\pi}^{\pi} F(t)\, P'_r(t-x)\, dt,$$

where $'$ denotes differentiation with respect to t; and, since P'_r is odd,

$$\frac{\partial F(r,x)}{\partial x} = -\frac{1}{\pi}\int_{0}^{\pi} \chi(t)\, 2 \sin t\, P'_r(t)\, dt,$$

where $\chi(t) = [F(x+t) - F(x-t)]/2\sin t$. Then, in order to prove the theorem, it is sufficient to show that the even function $-\sin t\, P'_r(t)/r = (1-r^2)\sin^2 t/\Delta_r^2(t)$ possesses the properties of positive kernels. Conditions (i) and (iii) of § 3.201 are obviously satisfied, and we verify (ii) by substituting $x = 0$, $F(t) = \sin t$, i. e. $\chi(t) = 1$.

3.441. *If $F'(x_0)$ exists and is finite, then $\partial F(r,x)/\partial x \to F'(x_0)$ when $(r,x) \to (1, x_0)$ along any path L not touching the circle.* Suppose, for simplicity, that $x_0 = 0$, $F(0) = 0$, and let $r = r(u)$, $x = x(u)$, $0 \leqslant u \leqslant 1$, $r(1) = 1$, be a parametric equation of L. Put $-\sin t\, P'_r(t-x) = A_u(t)$ for $(r,x) \in L$. The theorem will be proved, when we show that $A_u(t)$ satisfies the following conditions

(i) $\displaystyle\int_{-\pi}^{\pi} |A_u(t)|\, dt = O(1)$, (ii) $\displaystyle\frac{1}{\pi}\int_{-\pi}^{\pi} A_u(t)\, dt \to 1$,

(iii) $M_u(\delta) = \operatorname{Max} |A_u(t)|\ (0 < \delta \leqslant t \leqslant \pi)$ tends to 0, as $u \to 1$, i. e. that $A_u(t)$ is, essentially, a quasi-positive kernel. In fact, put-

[1]) l_1 and l_2 are contained between the smallest and the largest of the four derivates of F at the point x.

ting $F(t)/\sin t - F'(0) = G(t)$, and denoting by $\theta(u)$ the left-hand side of (ii), we deduce that

$$\frac{\partial F(r,x)}{\partial x} - \theta(u)\,F'(0) = \frac{1}{\pi}\int_{-\pi}^{\pi} G(t)\,A_u(t)\,dt = \int_{-\delta}^{\delta} + \int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}.$$

The last two integrals on the right tend to 0 for fixed δ (cond. (iii)), and the preceding term is small with δ (cond. (i)).

Now relation (ii) follows from the second formula 3.44(2) if we put $F(t) = \sin t$. The left-hand side of (i) is equal to

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |\sin(t+x)\,P_r'(t)|\,dt \leqslant 2\,|\sin x|\int_0^{\pi} |P_r'(t)|\,dt + 2\int_0^{\pi} \sin t\,|P_r'(t)|\,dt.$$

Since $P_r'(t) \leqslant 0$ in $(0, \pi)$, the first term on the right is less than $2x\,P_r(0) \leqslant 2x/(1-r) = O(1)$, if $(r, x) \in L$. The last term on the right is also bounded, $-2\sin t\,P_r'(t)/r$ being a positive kernel. Condition (iii) is obvious.

3.442. Corollary. Let F be an integral of f. *For any x_0 where $f(x_0)$ is finite and equal to $F'(x_0)$, we have $f(r,x) \to f(x_0)$, as $(r,x) \to (1, x_0)$ along any path not touching the circle.* In fact, supposing for simplicity that the constant term of $\mathfrak{S}[f]$ vanishes, we have $\mathfrak{S}[f] = \mathfrak{S}'[F]$, and the result follows from Theorem 3.441.

3.45. *At any point x where f is finite and is the differential coefficient of its integral F, we have*

$$(1)\qquad \bar f(r,x) - \left(-\frac{1}{\pi}\int_{\eta}^{\pi}\frac{f(x+t)-f(x-t)}{2\operatorname{tg}\frac{1}{2}t}\,dt\right) \to 0, \quad as\ r \to 1\ ^{1)},$$

where the number $\eta = \eta(r)$, $0 < \eta < \pi/2$, is the root of the equation $\cos x = 2r/(1+r^2)$. It is plain that $\eta \to 0$ as $r \to 1$. More precisely, from the formula $\sin\eta = (1-r)(1+r)/(1+r^2)$ we find that $\eta \simeq 1 - r^{2)}$.

The last formula in 3.4(1) gives us $\pi\bar f(r,x) = -\int_0^{\eta}\psi_x(t)\,Q_r(t)\,dt +$ $-\int_{\eta}^{\pi}\psi_x(t)\,[Q_r(t) - Q_1(t)]\,dt - \int_{\eta}^{\pi}\psi_x(t)\,Q_1(t)\,dt$, and we have only

[1] Privaloff [2], Plessner [2]. See also Fatou [1].

[2] The theorem holds true if we replace η by $1-r$ in (1), but this is irrelevant for our purposes.

to show that the first two terms on the right are $o(1)$. Let $B(h) = o(h)$ be the integral of $\psi_x(t)$ over the range $0 \leqslant t \leqslant h$. From the formula for $Q_r'(t)$ we see that $Q_r(t)$ is monotonically increasing in $(0, \eta)$. Hence, applying the second mean-value theorem to the first term, we find that it is equal to $Q_r(\eta)\,[B(\eta) - B(\tau)] = o(1)$, since $0 < \tau < r$ and $Q_r(\eta) < r + r^2 + \ldots < 1/(1-r)$. It is easy to verify that $Q_r(t) - Q_1(t) = -(1-r)^2\, Q_r(t)/2\,(1-\cos t)$. Applying the same mean-value theorem to the second term in question, we find that it is equal to the expression $(1-r)^2\, Q_r(\eta)/2 = O(1-r)$ multiplied by the integral

$$(2)\quad \int_\eta^\xi \frac{\psi_x(t)}{1-\cos t}\,dt = \left[\frac{B(t)}{1-\cos t}\right]_\eta^\xi + \int_\eta^\xi \frac{B(t)\sin t}{(1-\cos t)^2}\,dt \quad (\eta < \xi < \pi).$$

Since $B(\xi)/(1-\cos\xi) = o(\xi^{-1}) = o(\eta^{-1})$, and the last integrand is $o(t^{-2})$, the left-hand side of (2) is $o(\eta^{-1}) = o(1-r)^{-1}$ and this completes the proof.

Since $\eta(r)$ tends continuously to 0 as $r \to 1$, we see that *a necessary and sufficient condition for the summability A of $\bar{\mathfrak{S}}[f]$ at the point x, is the existence of the integral* 3.321(1), *which represents then the sum of $\bar{\mathfrak{S}}[f]$.*

3.5. The Cesàro summation of differentiated series. According to Theorem 3.442, $\mathfrak{S}[f]$ is summable A at any point x where f is the finite derivative of its integral, whereas to prove the summability $(C, 1)$ we used a somewhat stronger condition, viz. $\Phi_x(t) = o(t)$. Indeed it may be shown that the former condition does not ensure the summability $(C, 1)$ of $\mathfrak{S}[f]$. We will now prove that.

(i) *At every point x where $F'(x) = \lim\,[F(x+h) - F(x-h)]/2h$ exists and is finite, $\mathfrak{S}'(F)$ is summable (C, r), $r > 1$, to the value $F'(x)$.* (ii) *At every point x where f is finite and is the differential coefficient of its integral, $\mathfrak{S}[f]$ is summable (C, r), $r > 1$, to the value $f(x)$* [1].

To prove (i), of which (ii) is a corollary, it is sufficient to show that $L_n^r(t) = \sin t\,[K_n^r(t)]'$ is a quasi-positive kernel if $r > 1$ [2]. This will be a consequence of the inequalities

---

[1]) Lebesgue [3] for $r = 2$, Privaloff [2], Young [6] in the general case.

[2]) The situation is the same as in § 3.44, except that $\sin t\, F_r'(t)$ is a positive kernel.

(1) $|L_n^r(t)| \leqslant n$ for $0 \leqslant t \leqslant 1/n$, (2) $|L_n^r(t)| \leqslant C/n^{r-1}t^r$ for $1/n \leqslant t \leqslant \pi$,

valid for $1 < r < 2$. C is a constant independent of n and t.

Let D_k be Dirichlet's kernel. Since $|D_k'| \leqslant 1 + 2 + \dots + k \leqslant n^2$ for $0 \leqslant k \leqslant n$, we find that $|[K_n^r]'| \leqslant n^2$, i. e. $|L_n^r(t)| \leqslant n^2 t \leqslant n$ for $0 \leqslant t \leqslant 1/n$, and the inequality (1) is established.

Using Abel's transformation, we verify the formula $\sum_{k=0}^{n} A_k^\alpha e^{ikt} = \left[-A_n^\alpha e^{i(n+1)t} + \sum_{k=0}^{n} A_k^{\alpha-1} e^{ikt}\right] / (1-e^{it})$. Applying this formula twice to the last expression but one in 3.3(3), we find that

$$K_n^r(t) = C_n(2\sin\tfrac{1}{2}t)^{-2} + \Im\left\{\frac{e^{i(n+1/2)t}}{2A_n^r\sin\frac{1}{2}t}\left[(1-e^{-it})^{-r} - \frac{\sum\limits_{k=n+1}^{\infty} A_k^{r-3}e^{-ikt}}{(1-e^{-it})^2}\right]\right\} =$$

$$= C_n(2\sin\tfrac{1}{2}t)^{-2} + \frac{\sin\left[(n+\frac{1}{2}+\frac{1}{2}r)t - \frac{\pi}{2}r\right]}{A_n^r(2\sin\frac{1}{2}t)^{r+1}} + \Im\left[\frac{e^{i(n+1/2)t}}{A_n^r(2\sin\frac{1}{2}t)^3}\sum_{k=n+1}^{\infty} A_k^{r-3}e^{-ikt}\right],$$

where $C_n = A_n^{r-1}/A_n^r + A^{r-2}/2A_n^r = O(1/n)$. Let P, Q, R denote the three terms in the last formula for K_n^r. Then $P_n' = O(1/nt^3)$, $Q_n' = O(1/n^r t^{r+2}) + O(1/n^{r-1}t^{r+1}) = O(1/n^{r-1}t^{r+1})$ if $nt \geqslant 1$. Let $\alpha(t) = \exp i(n+3/2)t/(2\sin\frac{1}{2}t)^3$ and let $\beta(t)$ be the sum following $\alpha(t)$ in R. Using Theorem 1.22, we see that $|\beta(t)| \leqslant 4|A_{n+1}^{r-3}|/|1-e^{-it}| = O(n^{r-3}/t)$, $|\beta'(t)|$[1] $\leqslant 4(n+1)|A_{n+1}^{r-3}|/|1-e^{-it}| = O(n^{r-2}/t)$. Since, on the other hand, $\alpha(t) = O(t^{-3})$, $\alpha'(t) = O(n/t^3) + O(1/t^4) = O(n/t^3)$ if $nt \geqslant 1$, we find that $|R_n'| \leqslant |\alpha'\beta + \beta'\alpha|/A_n^r = O(n^{-2}t^{-4})$.

Collecting the results, we obtain that $[K_n^r(t)]' = O(1/nt^3) + O(1/n^{r-1}t^{r+1}) + O(n^{-2}t^{-4}) = O(1/n^{r-1}t^{r+1})$ if $nt \geqslant 1$. Thence we have $L_n^r(t) = O(n^{1-r}t^{-r})$ if $t \geqslant 1/n$, $1 < r < 2$, and this completes the proof.

Let $G(h)$ be the integral of $\varphi(t) = f(x+t) + f(x-t) - 2f(x)$ over the interval $0 \leqslant t \leqslant h$. Applying (ii) to $\mathfrak{S}[\varphi]$, we see that *$\mathfrak{S}[f]$ is summable (C, r), $r > 1$ at the point x and has the sum $f(x)$, if $G(h) = o(h)$ as $h \to 0$.*

Essentially the same proof shows that *under the hypothesis of Theorem* (ii), *we have the relation* 3.32(1), *for* $1 < r < 2$.

[1]) The series defining β' converges for $t \neq 0$ if $r < 2$.

3.6. Fourier sine series. Let $f(x)$ be an odd function. From the first formula 3.4(1) we deduce that

$$(1) \qquad f(r, x) = \frac{1}{\pi} \int_0^{\pi} f(t) \left[P_r(x-t) - P_r(x+t)\right] dt.$$

(i) *If* $f(x) \not\equiv 0$ *is odd and non-negative in* $(0, \pi)$, *the function* $f(r, x)$ *is positive for* $0 < x < \pi$. *More generally, if* $f(x) \not\equiv$ const. *satisfies an inequality* $m \leqslant f(x) \leqslant M$ *for* $0 < x < \pi$, *then*

$$(2) \quad m\mu(r, x) < f(r, x) < M\mu(r, x) \quad \text{for } 0 < x < \pi, \quad 0 < r < 1,$$

where $\mu(r, x)$, *which is positive for* $0 < x < \pi$, *is the Poisson integral for the function* $\mu(x) = \operatorname{sign} x$ $(|x| < \pi)$.

The first part of the theorem follows from (1) if we note that $P_r(x-t) > P_r(x+t)$ for $0 < x < \pi$, $0 < t < \pi$. For this reason we have also

$$\frac{m}{\pi}\int_0^{\pi}[P_r(x-t) - P_r(x+t)]\,dt < f(r, x) < \frac{M}{\pi}\int_0^{\pi}[P_r(x-t) - P_r(x+t)]\,dt,$$

which is just (2).

(ii) *Theorem* (i) *remains true if we replace summability A by summability* $(C, 3)$ [1]. In particular, the inequality (2) should be replaced by $m\mu_n^3(x) < \sigma_n^3(x) < M\mu_n^3(x)$, where σ_n^3 and μ_n^3 denote the $(C, 3)$ means of $\mathfrak{S}[f]$ and $\mathfrak{S}[\mu]$.

For the proof it is sufficient to show that the kernel $K_n^3(t)$ is a strictly decreasing function in $(0, \pi)$, or, $K_n^3(t)$ being a trigonometrical polynomial, that $[K_n^3(t)]' \leqslant 0$ in $(0, \pi)$. The last expression is the Cesàro mean $s_n^3(t)/A_n^3$ of the series $\frac{1}{2} + \cos t + \cos 2t + \ldots$ differentiated term by term. Thus from 3.11(1) we conclude that

$$\sum_{n=0}^{\infty} s_n^3(t)\, r^n = -\left[\frac{1-r^2}{2\,(1-r)^2\,\Delta_r(t)}\right]^2 \cdot \frac{4r\sin t}{1-r^2},$$

where $\Delta_r(t) = 1 - 2r\cos t + r^2$. Using the formulae 3.11(1) again, we see that the expression in square brackets is the power series $K_0(t) + 2K_1(t)\,r + \ldots + (n+1)\,K_n(t)\,r^n + \ldots$, where the coefficients $K_n(t) \geqslant 0$ are Fejér's kernels. Since $r/(1-r^2) = r + r^3 + \ldots$ has also

[1]) Fejér [4].

non-negative coefficients, we see that $s_n^3(t) \leqslant 0$ [1] in $(0, \pi)$, and this completes the proof.

3.7. Convergence factors. A sequence $\lambda_0, \lambda_1, \ldots$ is said to be *convex* if $\Delta^2\lambda_n \geqslant 0$, $n = 0, 1, \ldots$, where $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$, $\Delta^2\lambda_n = \Delta\lambda_n - \Delta\lambda_{n+1}$. Suppose, in addition, that $\{\lambda_n\}$ is bounded. Since, for $\{\lambda_n\}$ convex, $\Delta\lambda_n$ is non-increasing and cannot be negative for any value of n (for otherwise we should have $\lambda_n \to \infty$), we have $\Delta\lambda_n \geqslant 0$, i. e. $\lambda_n \geqslant \lambda_{n+1} \to \lambda > -\infty$. In the equation $\lambda_0 - \lambda = \Delta\lambda_0 + \Delta\lambda_1 + \ldots$ the terms on the right are steadily decreasing, and so, by a well-known theorem of Abel, $n\Delta\lambda_n \to 0$. Taking this into account, and applying to the series $1.\Delta\lambda_0 + 1.\Delta\lambda_1 + \ldots$ Abel's transformation, we obtain: *If $\{\lambda_n\}$ is convex and bounded, then $\{\lambda_n\}$ decreases, $n\Delta\lambda_n \to 0$ and the series*

$$(1) \qquad \sum_{n=0}^{\infty} (n+1)\, \Delta^2\lambda_n$$

converges to the sum $\lambda_0 - \lim \lambda_n$.

If a function $\lambda(x)$ is twice differentiable and $\lambda''(x) \geqslant 0$, the sequence $\{\lambda_n\} = \{\lambda(n)\}$ is convex. In fact, by the mean-value theorem, $\Delta^2\lambda_n = \Delta\lambda_n - \Delta\lambda_{n+1} = -\lambda'(\theta_n) + \lambda'(\theta_{n+1}) \geqslant 0$, where $n < \theta_n < n+1$. In particular, if we put $\lambda_n = 1/\log n$ for $n = 2, 3, \ldots$, and choose for λ_0, λ_1 suitable values, $\{\lambda_n\}$ will be convex.

We need the following lemma:

Let s_n and σ_n denote the partial sums and the first aritmetic means of a series $u_0 + u_1 + \ldots$ If $\{\sigma_n\}$ converges and $s_n = o(1/\mu_n)$, where $\{\mu_n\}$ is convex and tends to 0, the series $u_0\mu_0 + u_1\mu_1 + \ldots$ converges. Applying twice Abel's transformation to the partial sum t_n of the last series, we find that it is equal to

$$\sum_{k=0}^{n-2} (k+1)\,\sigma_k \Delta^2\mu_k + n\sigma_{n-1}\Delta\mu_{n-1} + s_n\mu_n \to \sum_{k=0}^{\infty} (k+1)\,\sigma_k\, \Delta^2\mu_k.$$

Remark. A sequence $\{\lambda_n\}$ will be called a *quasi-convex* sequence if the series (1) converges absolutely. The lemma will subsist for quasi-convex $\{\mu_n\}$ if we prove that $n\Delta\mu_{n-1} \to 0$. But

$$|\Delta\mu_{n-1}| = \lim_{N\to\infty} \Big| \sum_{k=n-1}^{N} \Delta^2\mu_k \Big| \leqslant n^{-1} \sum_{k=n-1}^{\infty} (k+1)\,|\Delta^2\mu_k| = o(n^{-1}).$$

[1] The first two coefficients of the series $[K_0 + 2K_1 r + \ldots]^2 = {}^1/_4 + 2K_1 r + \ldots$ are positive for $0 < t < \pi$, and this shows that $s_n^3(t) < 0$ for $0 < t < \pi$.

As a corollary we have the following theorem.

3.71. *If a_n, b_n are the Fourier coefficients of a function f, the series*

$$\sum_{k=2}^{\infty} \frac{a_k \cos kx + b_k \sin kx}{\log k}, \qquad \sum_{k=2}^{\infty} \frac{a_k \sin kx - b_k \cos kx}{\log k}$$

converge almost everywhere [1]) (§§ 2.73, 3.31, 3.321).

It is not difficult to deduce that *if f is continuous in (a, b), the first series converges uniformly in every interval $(a+\delta, b-\delta)$.*

3.8. Summability of Fourier-Stieltjes series [2]). Let $F(x)$, $0 \leqslant x \leqslant 2\pi$, be a function of bounded variation. From Theorems 2.13 and 3.5 we see that $\mathfrak{S}[dF]$ is summable (C, r), $r > 1$, at almost every point and has the sum $F'(x)$. We will now prove a stronger result, viz.

Let $\sigma_n^r(x)$ and $\bar{\sigma}_n^r(x)$ denote the r-th Cesàro means of $\mathfrak{S}[dF]$ and $\bar{\mathfrak{S}}[dF]$. If $0 < r \leqslant 1$, then

$$\text{(1a)} \quad \sigma_n^r(x) \to F'(x), \qquad \text{(1b)} \quad \bar{\sigma}_n^r(x) - \left\{ -\frac{1}{\pi} \int_{1/n}^{\pi} \frac{F(x+t) + F(x-t) - 2F(x)}{4 \sin^2 \frac{1}{2} t} \right\} dt \to 0,$$

for almost every x.

We shall only sketch the proof, which is similar to that of Theorems 3.31 and 3.32. First of all we need the following lemma, analogous to the result of § 2.703. *Let*

$$F_x^*(t) = F(x+t) - F(x-t) - 2tF'(x), \quad G_x^*(t) = F(x+t) + F(x-t) - 2F(x),$$

and let $\Phi_x^*(h)$, $\Psi_x^*(h)$ be the total variations of the functions $F_x^*(t)$, $G_x^*(t)$ over the interval $0 \leqslant t \leqslant h$. Then, for almost every x we have $\Phi_x^*(h) = o(h)$, $\Psi_x^*(h) = o(h)$.

Let α be an arbitrary number, and let $V_\alpha(t)$ be the total variation of the function $F(t) - \alpha t$. For almost every x we have $V'_\alpha(x) = |F'(x) - \alpha|$, i. e.

$$\frac{1}{h} \int_0^h |d_t\{F(x \pm t) - \alpha t\}| \to |F'(x) - \alpha| \quad \text{as } h \to +0,$$

where the suffix t indicates that the variation is taken with respect to the variable t. Considering rational values of α and arguing as in § 2.703, we prove that, for almost every x, we have

$$\text{(2)} \qquad \int_0^h |d_t\{F(x \pm t) - t F'(x)\}| = o(h),$$

and hence

$$\int_0^h |d_t F_x^*(t)| = o(h), \qquad \int_0^h |d_t G_x^*(t)| = o(h).$$

---

[1]) For the first part see Hardy [2], for the second Plessner [2].

[2]) Young [3], M. Riesz [2], Plessner [2].

Now it is easy to prove the theorem. From the formulae 1.47(1) we obtain that

$$\sigma_n^r = \frac{1}{\pi}\int_{-\pi}^{\pi} K_n^r(x-t)\,dF(t) = \frac{1}{\pi}\int_0^{\pi} K_n^r(t)\,d_t\{F(x+t)-F(x-t)\},$$

$$\sigma_n^r(x) - F'(x) = \frac{1}{\pi}\int_0^{\pi} K_n^r(t)\,d_t F_x^*(t);\quad |\sigma_n^r(x) - F(x)| \leqslant \frac{1}{\pi}\int_0^{\pi} |K_n^r(t)|\,|d_t F_x^*(t)| =$$

$$= \frac{1}{\pi}\int_0^{1/n} + \frac{1}{\pi}\int_{1/n}^{\pi}.$$

Supposing that $\Phi_x^*(h) = o(h)$, we obtain, in virtue of the inequalities 3.3(2), that the first term in the last sum is $\leqslant 2n\,\Phi_x^*(1/n)/\pi = o(1)$. Integrating by parts, we find that the second term does not exceed

$$\frac{C}{\pi n^r}[\Phi_x^*(t)\,t^{-r-1}]_{1/n}^{\pi} + \frac{C(1+r)}{\pi n^r}\int_{1/n}^{\pi}\Phi_x^*(t)\,t^{-r-2}\,dt = o(1),$$

and this gives the first part of the theorem. To obtain the second we observe that

$$\bar{\sigma}_n^r(x) = -\frac{1}{\pi}\int_0^{\pi}\bar{K}_n^r(t)\,d_t[F(x+t)+F(x-t)] = -\frac{1}{\pi}\int_0^{\pi}\bar{K}_n^r(t)\,d_t\,G_x(t),$$

$$\bar{\sigma}_n^r(x) - \left(-\frac{1}{\pi}\int_{1/n}^{\pi}\frac{d_t\,G_x(t)}{2\,\mathrm{tg}\,\frac{1}{2}t}\right) = -\frac{1}{\pi}\int_0^{1/n}\bar{K}_n^r(t)\,d_t\,G_x(t) + \frac{1}{\pi}\int_{1/n}^{\pi}H_n^r(t)\,d_t\,G_x(t)$$

(§ 3.32). From the lemma we easily deduce that each of the terms on the right is $o(1)$. Integrating by parts we verify that the left-hand side of the last equation differs from the left-hand side of (1b) by a term tending to 0 as $n\to\infty$. This completes the proof.

3.81. The lemma proved in the preceding section is of fundamental importance for the theory of Fourier-Stieltjes series. From it we deduce that the partial sums of $\mathfrak{S}[dF]$ and $\bar{\mathfrak{S}}[dF]$ are $o(\log n)$ at almost every point. Similarly, taking for granted the result that $\bar{\mathfrak{S}}[dF]$ is summable $(C,1)$ almost everywhere, we verify that Theorem 3.71 holds true for Fourier-Stieltjes series.

3.9. Miscellaneous theorems and examples.

1. Let (L) $x=\varphi(t)$, $y=\psi(t)$, $0\leqslant t\leqslant 2\pi$, be a closed and convex curve. If $\varphi_n(t)$ and $\psi_n(t)$ are the Fejér means of $\mathfrak{S}[\varphi]$ and $\mathfrak{S}[\psi]$, the curves $x=\varphi_n(t)$, $y=\psi_n(t)$, $n=0,1,\dots$, lie in the interior of the region limited by L. Fejér [5].

[If A, B, C are constants, and $A\varphi(t)+B\psi(t)+C\geqslant 0$, then $A\varphi_n(t)+B\psi_n(t)+C>0$].

2. Let $f_n(r,x)$ be the n-th partial sum of the series $f(r,x)$ (§ 3.4). If $m\leqslant f(x)\leqslant M$, $0\leqslant x\leqslant 2\pi$, then $m\leqslant f_n(r,x)\leqslant M$ for $0\leqslant r\leqslant\frac{1}{2}$, but not necessarily for $r>\frac{1}{2}$. Fejér [2].

[The expression $\frac{1}{2}+r\cos x+\ldots+r^n\cos nx=\dfrac{1-r^2-2r^{n+1}[\cos(n+1)x-r\cos nx]}{2(1-2r\cos x+r^2)}$ is non-negative for $r\leqslant 1/2$. The sum $\frac{1}{2}+r\cos x$ is negative for $x=\pi$, if $r>\frac{1}{2}$].

3. Let $F_x(h)$ and $\Phi_x(h)$ denote the integrals of $\varphi_x(t)$ and $|\varphi_x(t)|$ over the interval $0\leqslant t\leqslant h$. Neither of the conditions (i) $F_x(h)=o(h)$, (ii) $\Phi_x(h)=O(h)$ necessitates the summability $(C,1)$ of $\mathfrak{S}[f]$ at the point x. Show that if *both* of them are satisfied, then $\mathfrak{S}[f]$ is summable $(C,1)$ at the point x, to the value $f(x)$.

[The argument is analogous to that of § 3.31, except that now we consider the integrals of $\varphi(t)K_n(t)$ over intervals $(0,k/n)$, $(k/n,\pi)$, where k is large but fixed. In virtue of (ii), the second integral is small with $1/k$. The Fejér kernel has a bounded number of maxima and minima in $(0,k/n)$, and so, employing the second mean-value theorem[1]) and the relation (i), we obtain that the first integral tends to 0.

This generalization of Theorem 3.31 is typical and many other theorems may be generalized in the same way. The theorem is due to Hardy and Littlewood [5]].

4. Let $\{\alpha_n\}$ be an arbitrary sequence of numbers such that $\alpha_n=O(1/n)$, and let $\sigma_n^r(x)$ be the r-th Cesàro means of $\mathfrak{S}[f]$, $r>0$. At any point x where $\Phi_x(h)=o(h)$, we have $\sigma_n^r(x+\alpha_n)\to f(x)$.

[This is an analogue of Theorem 3.441. The proof is similar to that of Theorem 3.3].

5. Let $s_n^*(x)$ be the modified partial sums of $\mathfrak{S}[f]$ (§ 2.3). A necessary and sufficient condition for the convergence of the series (S) $\sum_{k=1}^{\infty}\dfrac{s_k^*-f}{k}$ at a point x where $\Phi_x(h)=o(h)$, is the existence of the integral $\displaystyle\int_0^{\pi}\frac{\varphi_x(t)}{2\sin\frac{1}{2}t}\,dt$.

[Let $u_n(x)$ be the n-th partial sum of the series $\sin x+\frac{1}{2}\sin 2x+\ldots=(\pi-x)/2$, $r_n(x)=(\pi-x)/2-u_n(x)$. It is plain that $|u_n(x)|\leqslant nx$, and making Abel's transformation we obtain that $r_n(x)=O(1/nx)$. Let $S_n(x)$ be the n-th partial sum of S. We have

$$S_n(x)=\frac{1}{\pi}\int_0^{\pi}\frac{\varphi_x(t)}{2\sin\frac{1}{2}t}u_n(t)\,dt=\frac{1}{\pi}\int_0^{1/n}+\frac{1}{\pi}\int_{1/n}^{\pi}=A+B.$$

Now $A\to 0$, and, in virtue of the inequality for r_n, we obtain that $S_n(x)-\dfrac{1}{\pi}\displaystyle\int_{1/n}^{\pi}\frac{\varphi_x(t)}{2\sin\frac{1}{2}t}\,\frac{\pi-t}{2}\,dt\to 0$. See also Hardy and Littlewood [4]].

6. Let $s_n(x)$ be the n-th partial sum of $\mathfrak{S}[f]$. If $f\in\mathrm{Lip}\,\alpha$, $0<\alpha\leqslant 1$, then $|s_n(x)-f(x)|=O(n^{-\alpha}\log n)$, (Lebesgue [1]).

---

[1]) Instead of this we may integrate by parts. The latter argument holds true for the method (C,r), $r>0$.

[See the expression 2.701(1), where the last term is now $O(n^{-\alpha-1})$. It has been shown by Lebesgue (l. c.) that the logarithm in the term $O(n^{-\alpha}\log n)$ cannot be omitted].

7. Let $\sigma_n(x)$ be the first arithmetic means of $\mathfrak{S}[f]$. If $f\in \mathrm{Lip}\,\alpha$, $0<\alpha<1$, then $\sigma_n(x)-f(x)=O(n^{-\alpha})$. If $\alpha=1$, then $\sigma_n(x)-f(x)=O(\log n)/n$. S. Bernstein [1]

$$[\pi|\sigma_n-f|\leqslant\int_0^\pi|\varphi_x(t)|\,K_n(t)\,dt\leqslant n\int_0^{1/n}O(t^\alpha)\,dt+\frac{1}{n}\int_{1/n}^\pi O(t^{\alpha-2})\,dt].$$

8. That the previous theorem cannot be strenghtened for $\alpha=1$, may be seen from the following result. If at a point x the right-hand side and the left-hand side derivatives exist, and $f'(x+0)-f'(x-0)=g$, then we have $\sigma_n(x)-f(x)\simeq 2g(\log n)/\pi n$. Szàsz [1], Alexits [1], Jacob [1].

[Let $g=1$. We have then $\varphi(t)=4(1+\varepsilon(t))\sin\frac{1}{2}t$, where $\varepsilon(t)=o(1)$.

$$\sigma_n(x)-f(x)=\frac{1}{\pi(n+1)}\int_0^\pi\frac{1-\cos(n+1)t}{\sin\frac{1}{2}t}\,dt+\frac{1}{\pi(n+1)}\int_0^\pi\varepsilon(t)\frac{1-\cos(n+1)t}{\sin\frac{1}{2}t}\,dt.$$

The first term on the right is $\simeq 2(\log n)/\pi n$, and the second is $o(\log n)/n$ (§ 2.631)].

9. If f is integrable in the sense of Denjoy-Perron, then, for almost every x, $\mathfrak{S}[F]$ is summable (C,r), $r>1$, to the value $f(x)$. Privaloff [1].

[This is a corollary of Theorem 3.5].

10. If $l=f(x+0)-f(x-0)$ exists and is finite, the sequence $nb_n(x)=$ $=n(b_n\cos nx-a_n\sin nx)$ is summable (C,r), $r>1$, to the value l/π. If f is of bounded variation, the theorem holds true for $r>0$. Fejér [3].

[The proof of the first part is similar to that of Theorem 3.5].

11. The sequence $\{s_\nu\}$ is said to be summable by the first logarithmic mean, to the value s, if $\tau_n=(s_1+s_2/2+\ldots+s_n/n)/\log n\to s$ as $n\to\infty$. If $\{s_\nu\}$ is summable $(C,1)$ to s, then $\tau_n\to s$.

[For the theory of the logarithmic means see Hardy and Riesz, *Dirichlet's series*].

12. The method considered in the previous problem may be sometimes effective if the sequence is summable $(C,1+\varepsilon)$ for any $\varepsilon>0$, but not for $\varepsilon=0$. An instance in point is Theorem 2.631, which may be interpreted in the sense that the sequence $n(a_n\sin nx-b_n\cos nx)$ is summable by the first logarithmic mean (see also § 3.9.10). Theorem 3.5 may be completed in the same way: If $F'(x)=\lim[F(x+h)-F(x-h)]/2h$ exists and is finite, then $\mathfrak{S}'[F]$ is summable at the point x by the first logarithmic mean and has the sum $F'(x)$. Zygmund [1], Hardy [6].

13. Let f be integrable, $\varphi_x(t)=f(x+t)+f(x-t)-2f(x)$. The summability of $\bar{\mathfrak{S}}'[f]$ is closely connected with the existence of $\lim\limits_{h\to 0}I(x,h)$, where

$I(x,h) = -\frac{1}{4\pi}\int_h^{\pi} \frac{\varphi_x(t)}{\sin^2 \frac{1}{2}t}\,dt$. More precisely, at any point x where (*) $\int_0^h \varphi_x(t)\,dt = o(h^2)$ (in particular at any point where $f'(x)$ exists and is finite) we have relation $\frac{\partial \bar{f}(r,x)}{\partial x} - I(x, 1-r) \to 0$ as $r \to 1$ Plessner [2].

14. A result analogous to the previous theorem holds for Cesàro means of order $r > 1$, or for the first logarithmic mean[1]). The proof is similar to that of Theorem 3.5.

15. In Theorem 3.6 (ii), summability $(C, 3)$ cannot be replaced by summability $(C, 2)$. Fejér [4].

[$\{K_n^2(t)\}'$ is positive, if $\sin(n+{}^3/_2)t = 0$, $\cos(n+{}^3/_2)\,t = -1$, $\cos \frac{1}{2}t < \frac{1}{3}$].

---

[1]) In the condition (*), $\varphi_x(t)$ must be replaced by $|\varphi_x(t)|$.

CHAPTER IV.

Classes of functions and Fourier series.

4.1. Inequalities. We begin by proving a number of inequalities which will be applied in the sequel [1].

Let $\varphi(u) \geqslant 0$ for $u \geqslant 0$. We say that $f(x)$, $a \leqslant x \leqslant b$, belongs to the class $L_\varphi(a, b)$ if the function $\varphi(|f|)$ is integrable over (a, b). If it is not necessary to specify the interval, we denote the class by L_φ simply. If $\varphi(u) = u^r$, $r > 0$, we write L^r instead of L_φ, L instead of L^1 and put

$$\mathfrak{M}_r[f;\ a, b] = \left(\int_a^b |f|^r dx\right)^{1/r}, \quad \mathfrak{A}_r[f;\ a, b] = \left(\frac{1}{b-a}\int_a^b |f|^r dx\right)^{1/r}.$$

When the interval (a, b) is fixed, we shall write simply $\mathfrak{M}_r[f]$, $\mathfrak{A}_r[f]$. The former expression may have a meaning even when (a, b) is infinite. If $r = 1$ we shall write $\mathfrak{M}$, $\mathfrak{A}$ instead of $\mathfrak{M}_1$, $\mathfrak{A}_1$.

Similarly, given a sequence $a = \{a_n\}$, finite or infinite, we write

$$\mathfrak{S}_r[a] = (\sum |a_n|^r)^{1/r}\ {}^{2)}.$$

4.11. Young's inequality. Let $\varphi(u)$, $u \geqslant 0$, $\psi(v)$, $v \geqslant 0$, be two functions, continuous, vanishing at the origin, strictly increasing,

---

[1]) For a detailled discussion of various inequalites see Hardy, Littlewood and Pólya, *Inequalities.*

[2]) Given a *finite* sequence $a = a_1, a_2, \dots, a_N$, let $\mathfrak{B}_r[a] = \left(\frac{1}{N}\sum_{n=1}^{N} |a_n|^r\right)^{1/r}$. This expression has properties analogous to those of $\mathfrak{A}_r[f]$, but we shall not consider it here.

tending to ∞, and inverse to each other. Then, for $a, b \geqslant 0$, we have the inequality, due to Young[1]),

(1) $$ab \leqslant \Phi(a) + \Psi(b), \text{ where } \Phi(x) = \int_0^x \varphi\, du,\ \Psi(y) = \int_0^y \psi\, dv.$$

The geometrical proof is obvious. It is also easy to see that the sign $\leqslant$ can be replaced by $=$ if and only if $b = \varphi(a)$. The functions Φ and Ψ will be called *complementary* functions. If $\varphi(u) = u^\alpha$, $\psi(v) = v^{1/\alpha}$, $\alpha > 0$, $1 + \alpha = r$, $1 + 1/\alpha = r'$, we obtain

(2) $$ab \leqslant \frac{a^r}{r} + \frac{b^{r'}}{r'},$$

where the 'complementary' exponents r, r' are connected by the relation $1/r + 1/r' = 1$[2]). This is a generalization of the well-known inequality $2ab \leqslant a^2 + b^2$, to which it reduces if $r = r' = 2$. It is plain that either $r \leqslant 2 \leqslant r'$ or $r' \leqslant 2 \leqslant r$. From (1) we deduce that, if $f(x) \in L_\Phi$, $g(x) \in L_\Psi$, the product fg is integrable. In particular, fg is integrable if $f \in L^r$, $g \in L^{r'}$.

4.12. Hölder's inequalities. Consider non-negative sequences $A = \{A_n\}$, $B = \{B_n\}$, $AB = \{A_n B_n\}$, and suppose that $\mathfrak{N}_r[A] = \mathfrak{N}_{r'}[B] = 1$, $r > 1$. Substituting, in 4.11(2), A_n, B_n for a, b, and adding all the inequalities, we obtain that $\mathfrak{N}[AB] \leqslant 1$. If $\{a_n\}$, $\{b_n\}$ are non-negative and $\mathfrak{N}_r[a]$, $\mathfrak{N}_{r'}[b]$ positive and finite, then, putting $A_n = a_n/\mathfrak{N}_r[a]$, $B_n = b_n/\mathfrak{N}_{r'}[b]$, we have $\mathfrak{N}_r[A] = 1$, $\mathfrak{N}_{r'}[B] = 1$, and from $\mathfrak{N}[AB] \leqslant 1$ we obtain the first of the Hölder inequalities

(1) $$\mathfrak{N}[ab] \leqslant \mathfrak{N}_r[a]\,\mathfrak{N}_{r'}[b], \qquad \mathfrak{M}[fg] \leqslant \mathfrak{M}_r[f]\,\mathfrak{M}_{r'}[g], \quad r > 1,$$

which is obviously true also if $\mathfrak{N}_r[a] = 0$ or $\mathfrak{N}_{r'}[b] = 0$. The second inequality (1), where $f, g \geqslant 0$, is obtained by the same argument, summation being replaced by integration. In the general case (a, b, f, g complex), we have

(2) $$\left|\sum a_n b_n\right| \leqslant \mathfrak{N}_r[a]\,\mathfrak{N}_{r'}[b], \quad \left|\int_a^b fg\, dx\right| \leqslant \mathfrak{M}_r[f]\,\mathfrak{M}_{r'}[g],$$

---

[1]) Young [7].

[2]). This notation will be used systematically in this chapter, so that by p' we shall denote the exponent q such that $1/p + 1/q = 1$.

since the left-hand sides in (2) do not exceed $\mathfrak{N}[ab]$, $\mathfrak{M}[fg]$ respectively.

A little attention shows that the first relation (2) degenerates into equality if and only if $\arg(a_n b_n)$ and $|a_n|^r/|b_n|^{r'}$ are independent of n ($\arg 0$ and $0/0$ denote any numbers we please). For the second relation the conditions are: $\arg f(x)\,g(x)$ and $|f(x)|^r/|g(x)|^{r'}$ must be constant almost everywhere.

The number M, finite or infinite, will be called the *essential upper bound* of a function $g(x)$, $a \leqslant x \leqslant b$, if (i) $g(x) \leqslant M$ almost everywhere, (ii) for every $M' < M$ the set of x for which $g(x) > M'$ is of positive measure. If $M < \infty$, we shall call f an *essentially bounded* function. We will prove that *if M is the essential upper bound of $|f(x)|$ in (a, b), then $\mathfrak{M}_r[f; a, b] \to M$ as $r \to \infty$*[1]. In the first place $\mathfrak{M}_r[f] \leqslant M(b-a)^{1/r}$, so that $\overline{\lim}\, \mathfrak{M}_r[f] \leqslant M$. Next, if M' is any number $< M$, and E the set of points where $|f(x)| > M'$, then $\mathfrak{M}_r[f] \geqslant |E|^{1/r} M'$, $\underline{\lim}\, \mathfrak{M}_r[f] \geqslant M'$, and so $\lim \mathfrak{M}_r[f] = M$. This completes the proof in the case of (a, b) finite, or when (a, b) is infinite and $M = \infty$. Let now (a, b) be infinite and $0 < M < \infty$. We may suppose that $M = 1$. The same agument as before proves that $\underline{\lim}\, \mathfrak{M}_r[f] \geqslant 1$. To show that $\lim \mathfrak{M}_r[f] = 1$ we need only observe that $\mathfrak{M}_r^r[f]$ is a decreasing function of r which, by the preceding remark, is $\geqslant 1$.

In virtue of the result just established, it is natural to define $\mathfrak{M}_\infty[f; a, b]$ as the essential upper bound of $|f|$ in (a, b). By L^∞ we may denote the class of essentially bounded functions. The second inequality (2) has then a meaning (and is obviously true) even when $r = \infty$.

Since any series $a_0 + a_1 + \dots$, $a_n \to 0$, can be represented as the integral, over $(0, \infty)$, of a function $f(x)$, where $f(x) = a_n$ for $n \leqslant x < n+1$, $n = 0, 1, \dots$, the above remarks apply also to series.

4.121. Let $f_i \in L^{r_i}$, $i = 1, 2, \dots, k$, where $r_i > 0$, $1/r_1 + 1/r_2 + \dots + 1/r_k = 1$. An easy induction shows that $\mathfrak{M}[f_1 f_2 \dots f_k] \leqslant \mathfrak{M}_{r_1}[f_1]\, \mathfrak{M}_{r_2}[f_2] \dots \mathfrak{M}_{r_k}[f_k]$. Similarly for series.

4.13. Minkowski's inequality. Let $a = \{a_n\}$, $b = \{b_n\}$ be two sequences, $a + b = \{a_n + b_n\}$. We will now prove Minkowski's inequality

---

[1]) Hence $\mathfrak{A}_r[f] \to M$ as $r \to \infty$.

(1) $$\mathfrak{N}_r[a+b] \leqslant \mathfrak{N}_r[a] + \mathfrak{N}_r[b], \qquad r \geqslant 1.$$

Writing $(a_n+b_n)^r = (a_n+b_n)^{r-1}a_n + (a_n+b_n)^{r-1}b_n$, and applying Hölder's inequality to the sums of terms $(a_n+b_n)^{r-1}a_n$ and of terms $(a_n+b_n)^{r-1}b_n$, we find that $\mathfrak{N}_r^r[a+b] \leqslant \mathfrak{N}_r^{r-1}[a+b]\,\mathfrak{N}_r[a] + \mathfrak{N}_r^{r-1}[a+b]\,\mathfrak{N}_r[b]$, and (1) follows.

The same argument proves Minkowski's inequality for integrals:

(2) $$\mathfrak{M}_r[f+g] \leqslant \mathfrak{M}_r[f] + \mathfrak{M}_r[g], \qquad r \geqslant 1.$$

If $0 < r < 1$, all these inequalities cease to be true. However we have then

(3) $$\mathfrak{M}_r^r[f+g] \leqslant \mathfrak{M}_r^r[f] + \mathfrak{M}_r^r[g], \quad \mathfrak{N}_r^r[a+b] \leqslant \mathfrak{N}_r^r[a] + \mathfrak{N}_r^r[b], \quad 0<r<1,$$

which inequalities are simple corollaries of the inequality $(x+y)^r \leqslant x^r + y^r$, $x \geqslant 0$, $y \geqslant 0$, $0 < r \leqslant 1$, or, what amounts to the same thing, of the inequality $(1+x)^r \leqslant 1 + x^r$. To prove the latter we observe that $(1+x)^r - 1 - x^r$ vanishes for $x = 0$ and has a negative derivative for $x > 0$ [1]).

Let $h(x, y)$ be a function defined for $a \leqslant x \leqslant b$, $c \leqslant y \leqslant d$. An argument similar to that which led to (2) gives the inequality

(4) $$\left\{\int_a^b \left|\int_c^d h(x,y)\,dy\right|^r dx\right\}^{1/r} \leqslant \int_c^d \left\{\int_a^b |h(x,y)|^r\,dx\right\}^{1/r} dy, \qquad r \geqslant 1,$$

which may be considered as the most general form of Minkowski's inequality since it contains the results (1) and (2) as special cases [2]).

4.14. Convex functions and Jensen's inequality. A function $\varphi(x)$, $\alpha \leqslant x \leqslant \beta$, is said to be *convex* if, for any pair of points P_1, P_2 on the curve $y = \varphi(x)$, the points of the arc $P_1 P_2$ are below, or on, the chord $P_1 P_2$. As an example we quote the function x^p, $p \geqslant 1$, which is convex in the interwal $(0, \infty)$.

For any system of positive numbers $p_1, p_2, \dots, p_n$, and any system of points $x_1, x_2, \dots, x_n$ from (α, β), we have the inequality

---

1) From the inequalities (2) and (3) we conclude that, if $f \in L^r$, $g \in L^r$, then $(f+g) \in L^r$, $r > 0$.

2) If $(c, d) = (0, 2)$, $h(x, y) = f(x)$ for $0 \leqslant y < 1$, $h(x, y) = g(x)$ for $1 \leqslant y \leqslant 2$, the inequality (4) reduces to (2). If $f(x) = a_n$, $g(x) = b_n$ for $n \leqslant x < n+1$, $n = 0, 1, \dots$, we obtain the inequality (1).

$$(1)\qquad \varphi\left(\frac{p_1 x_1 + p_2 x_2 + \ldots + p_n x_n}{p_1 + p_2 + \ldots + p_n}\right) \leqslant \frac{p_1 \varphi(x_1) + \ldots + p_n \varphi(x_n)}{p_1 + \ldots + p_n},$$

due to Jensen[1]. For $n=2$ this is just the definition of convexity, and for $n>2$ it follows by induction.

If is obvious geometrically that, if φ is convex, $\varphi(x+0)$, and similarly $\varphi(x-0)$, must exist. These limits can be neither $+\infty$ nor $-\infty$. Moreover $\varphi(x+0)=\varphi(x-0)=\varphi(x)$, i. e. convex functions are continuous.

Assuming φ continuous, we may take as the definition of convexity that for every arc $P_1 P_2$ there exists a subarc $P_1' P_2'$ lying below or on the chord $P_1' P_2'$. In fact, if there existed an arc $P_1 P_2$ lying, even partially, above the chord $P_1 P_2$, there would exist a subarc $P_1' P_2'$ lying totally above the chord $P_1' P_2'$, so that the two definitions of convexity are equivalent.

It is easy to see that a convex function has no proper maximum in the interior of the interval in which it is defined. Let $\varphi(x)$ be convex in $(0,\infty)$ and let x_0 be a minimum of φ. If $\varphi(x)$ is not constant for $x>x_0$, then $\varphi(x)$ tends to $+\infty$, as $x\to\infty$, at least as rapidly as a multiple of x. This follows from the fact that, if $x_0<x_1<x_2<\ldots$, $x_n\to\infty$, the angles which the chords joining $(x_i, \varphi(x_i))$ and $(x_{i+1}, \varphi(x_{i+1}))$ make with the real axis, increase with i. Therefore, if $\varphi(u)$ is convex in $(0,\infty)$, and $\varphi(u)\to\infty$ with u, the relation $f\in L_\varphi$ involves the integrability of f.

Let $f(t)$, $p(t)$ be two functions defined for $a\leqslant t\leqslant b$, and such that $\alpha\leqslant f(t)\leqslant\beta$, $p(t)\geqslant 0$, $p(t)\not\equiv 0$. Let $\varphi(u)$ be a convex function defined for $\alpha\leqslant u\leqslant\beta$. Jensen's inequality for integrals, viz.,

$$(2)\qquad \varphi\left\{\frac{\int_a^b f(t)\,p(t)\,dt}{\int_a^b p(t)\,dt}\right\} \leqslant \frac{\int_a^b \varphi(f)\,p(t)\,dt}{\int_a^b p(t)\,dt},$$

is a simple corollary of (1) if $f(t)$ and $p(t)$ are continuous and (a,b) is finite. In fact, if $a=t_0<t_1<\ldots<t_n=b$ is a subdivision of (a,b), $\delta_i=t_i-t_{i-1}$, $p_i=p(t_i)\,\delta_i$, $x_i=f(t_i)$, the inequality (1) tends to (2), provided that $\mathrm{Max}\,\delta_i\to 0$. To prove (2) in the most

[1]) Jensen [1].

general case is not a difficult task, but for the sake of brevity we content ourselves with the case which we shall actually need later, viz. $f \geqslant 0$, $\varphi(u)$ non-negative and increasing with u, (a, b) finite. Since any bounded f is the limit of a uniformly bounded sequence of continuous functions f_n,[1]) we obtain (2) for f and p bounded. Similarly, for f and p integrable, we have $f = \lim f_n$, $p = \lim p_n$, where each f_n and p_n is bounded and $f_n \leqslant f_{n+1}$, $p_n \leqslant p_{n+1}$; an application of Lebesgue's theorem on the integration of monotonic sequences yields the desired result.

4.141. *A necessary and sufficient condition that a function $\chi(x)$ defined at every point of an interval $\alpha \leqslant x \leqslant \beta$, $-\infty < \alpha < \beta < \infty$, should be convex, is that $\chi(x)$ should be the indefinite integral of a function non-decreasing and integrable over (α, β),* i. e.

(1) $$\chi(x) = \chi(\alpha) + \int_\alpha^x \xi(t)\,dt, \quad \text{where} \quad \xi(t_1) \leqslant \xi(t_2) \quad \text{for} \quad t_1 \leqslant t_2.$$

Suppose first that the condition (1) is satisfied. Since instead of (α, β) we may consider an arbitrary subinterval of (α, β), it is sufficient to show that, if $0 < \vartheta < 1$, $x = (1-\vartheta)\alpha + \vartheta\beta$, the function χ satisfies the inequality $\chi(x) \leqslant (1-\vartheta)\chi(\alpha) + \vartheta\chi(\beta)$. Without real loss of generality we may assume that $\alpha = 0$, $\chi(\alpha) = 0$, so that the inequality which we have to prove is

$$\int_0^{\vartheta\beta} \xi(t)\,dt \leqslant \vartheta \int_0^\beta \xi(t)\,dt, \quad \text{or} \quad (1-\vartheta)\int_0^{\vartheta\beta} \xi(t)\,dt \leqslant \vartheta \int_{\vartheta\beta}^\beta \xi(t)\,dt.$$

Now it is sufficient to observe that the left-hand side of the last inequality is at most equal to, and the right-hand side is not less than, the number $\vartheta(1-\vartheta)\beta\,\xi(\vartheta\beta)$.

To prove the second half of the theorem let $R(x, h)$ denote the ratio $[\chi(x+h) - \chi(x)]/h$, $h \neq 0$. From the convexity of χ it follows that

(2) $$R(x, -k) \leqslant R(x, h), \qquad (3) \quad R(x, h) \leqslant R(x, h_1)$$

provided that $0 < k$, $0 < h < h_1$, and that the points $x, x-k, x+h_1$ belong to (α, β). From (3) we see that $R(x, h)$ tends to a definite limit as $h \to +0$, and, in virtue of (2), this limit, which is the right-hand derivative $D^+\chi(x)$, is finite for $\alpha < x < \beta$. Similarly we prove that $R(x, -h_1) \leqslant R(x, -h)$ for $0 < h < h_1$, and that the left-hand derivative $D^-\chi(x)$ exists and is finite for $\alpha < x < \beta$. It follows from (2) that

(4) $$D^-\chi(x) \leqslant D^+\chi(x).$$

---

[1]) Let $F(x)$ be the indefinite integral of $f(x)$. We may put e. g. $f_n(x) = n\,[F(x + 1/n) - F(x)]$.

Let now $\alpha < x < x_1 < \beta$, and let $h > 0$, $k > 0$, $h + k = x_1 - x$, so that $x + h = x_1 - k$. We have then $D^+\chi(x) \leqslant R(x, h) \leqslant R(x_1, -k) \leqslant D^-\chi(x_1)$. From this and from the inequalities (4) we obtain that, for $x < x_1$,

$$D^-\chi(x) \leqslant D^-\chi(x_1), \quad D^+\chi(x) \leqslant D^+\chi(x_1), \tag{5}$$

i. e. the derivatives $D^-\chi(x)$ and $D^+\chi(x)$ are non-decreasing. Since the set of points where a non-decreasing function is discontinuous is at most enumerable, we infer from (4) and (5) that the set of points where $\chi'(x)$ does not exist is at most enumerable. The derivative $\chi'(x)$ is uniformly bounded in every interval (α', β') completely interior to (α, β). Hence the equation (1) is certainly true if we replace α by α', $\xi(t)$ by $\chi'(t)$ and suppose that $\alpha' \leqslant x \leqslant \beta'$. Making $\alpha' \to \alpha$, $\beta' \to \beta$, and remembering that $\chi(t)$ is continuous, we obtain the formula (1), with $\xi(t) = \chi'(t)$. To show that $\chi'(t)$ is integrable we need only observe that it is of constant sign in the neighbourhood of the points α and β, so that the existence of improper integrals involves the integrability in the sense of Lebesgue. This completes the proof.

4.142. Let now $\varphi(x)$ be an arbitrary function non-negative, non-decreasing, tending to ∞ with x, and vanishing at the origin. The curve $y = \varphi(x)$ may possess discontinuities and stretches of invariability. The inverse function $x = \psi(y)$ has the same properties, and is one valued except for the values which correspond to the stretches of invariability of $\varphi(x)$. If $\varphi(x)$ is constant and has a value y_0 for $\alpha < x < \beta$, we assign to $\psi(y_0)$ any value from the interval (α, β). Since the number of the stretches of invariability is at most enumerable, our choice has no influence upon the values of the integral $\Phi(x)$ of $\varphi(x)$, and it is easy to see that the Young inequality 4.11(1) holds true in this slightly more general case.

From the theorem proved in § 4.141 it follows that every function $\Phi(x)$ which is non-negative, convex, and satisfies the relations $\Phi(0) = 0$ and $\Phi(x)/x \to \infty$ as $x \to \infty$, may be considered as a Young function. More precisely to every such function $\Phi(x)$ corresponds another function $\Psi(x)$ with similar properties, and such that $ab \leqslant \Phi(a) + \Psi(b)$ for every $a \geqslant 0$, $b \geqslant 0$. It is sufficient to take for $\Psi(x)$ the integral of the function $\psi(x)$ inverse to the function $\varphi(x) = \Phi'(x)$. Since $\Phi(x)/x \to \infty$ with x, it is easy to see that $\varphi(x)$ and $\psi(x)$ are unbounded as $x \to \infty$.

4.15. $\mathfrak{M}_\alpha[f]$ and $\mathfrak{A}_\alpha[f]$ as functions of α. A function $\psi(u) \geqslant 0$ will be called a *multiplicatively convex* function if, for every $t_1 \geqslant 0$, $t_2 \geqslant 0$, $t_1 + t_2 = 1$, we have $\psi(t_1 u_1 + t_2 u_2) \leqslant \psi^{t_1}(u_1)\psi^{t_2}(u_2)$. It is the same thing as to say that $\log \psi(u)$ is convex.

Given a function $f(x)$, the expression $\mathfrak{A}_\alpha[f]$ is a non-decreasing function of α. $\mathfrak{A}_\alpha^\alpha[f]$ and $\mathfrak{M}_\alpha^\alpha[f]$ are multiplicatively convex functions of α $(\alpha > 0)$ [1].

---

[1] Hausdorff [2].

Substituting $|f|^\alpha$ for f, 1 for g, in the second formula 4.12(1), and dividing both sides by $b-a$, we obtain that $\mathfrak{A}_\alpha[f] \leqslant \mathfrak{A}_{\alpha r}[f]$ for $r > 1$. That the result is not true for $\mathfrak{M}_\alpha$, is easily seen from the example $a = 0$, $b = 2$, $f(x) = 1$.

To prove the second part of the theorem, let $\alpha = \alpha_1 t_1 + \alpha_2 t_2$, $\alpha_i > 0$, $t_i > 0$, $t_1 + t_2 = 1$. Replacing the integrand $|f|^\alpha$ by $|f|^{\alpha_1 t_1} |f|^{\alpha_2 t_2}$, in $\mathfrak{M}_\alpha$, and applying Hölder's inequality with $r = 1/t_1$, $r' = 1/t_2$, we find: $\mathfrak{M}_\alpha^\alpha \leqslant \mathfrak{M}_{\alpha_1}^{\alpha_1 t_1} \mathfrak{M}_{\alpha_2}^{\alpha_2 t_2}$. Dividing both sides by $b-a$, we obtain that $\mathfrak{A}_\alpha^\alpha \leqslant \mathfrak{A}_{\alpha_1}^{\alpha_1 t_1} \mathfrak{A}_{\alpha_2}^{\alpha_2 t_2}$.

4.16. A theorem of Young. Let $f(x)$ and $g(x)$ be two functions of period 2π, belonging to $L^p(0, 2\pi)$ and $L^q(0, 2\pi)$ respectively, and let

(1) $$h(x) = \int_0^{2\pi} f(x+t)\, g(t)\, dt.$$

Then, *if* $1/p + 1/q > 1$, *and* $1/r = 1/p + 1/q - 1$, *the function* $h(x)$ *is of the class* L^r *and, moreover,*

(2) $$\mathfrak{M}_r[h] \leqslant \mathfrak{M}_p[f]\, \mathfrak{M}_q[g].$$

We may suppose that $f \geqslant 0$, $g \geqslant 0$. Let λ, μ, ν be any three positive numbers such that $1/\lambda + 1/\mu + 1/\nu = 1$. Writing $f(x+t)\,g(t)$ in the form $f^{p/\lambda} g^{q/\lambda} f^{p(1/p-1/\lambda)} g^{q(1/q-1/\lambda)}$, and applying Hölder's inequality with the exponents λ, μ, ν (§ 4.121), we see that $h(x)$ does not exceed

$$\left[\int_0^{2\pi} f^p(x+t)\, g^q(t)\, dt\right]^{1/\lambda} \left[\int_0^{2\pi} f^{p\mu(1/p-1/\lambda)}(x+t)\, dt\right]^{1/\mu} \left[\int_0^{2\pi} g^{q\nu(1/q-1/\lambda)}(t)\, dt\right]^{1/\nu}.$$

If we suppose that $1/p - 1/\lambda = 1/\mu$, $1/q - 1/\lambda = 1/\nu$, $\lambda = r$, the condition $1/\lambda + 1/\mu + 1/\nu = 1$ involves $1/p + 1/q - 1/r = 1$. The last two factors in the product are equal to $\mathfrak{M}_p^{p/\mu}[f]\, \mathfrak{M}_q^{q/\nu}[g]$, and the result follows from the formula

$$\int_0^{2\pi} dx \left\{\int_0^{2\pi} f^p(x+t)\, g^q(t)\, dt\right\} = \mathfrak{M}_p^p[f]\, \mathfrak{M}_q^q[g]$$

(§ 2.12). We add two remarks:

(i) The inequality (2) may be stated in a slightly different form. If we put $p = 1/(1-\alpha)$, $q = 1/(1-\beta)$, $0 < \alpha < 1$, $0 < \beta < 1$, then $\mathfrak{M}_{\frac{1}{1-\gamma}}[h] \leqslant \mathfrak{M}_{\frac{1}{1-\alpha}}[f]\,\mathfrak{M}_{\frac{1}{1-\beta}}[g]$, where $\gamma = \alpha + \beta < 1$.

(ii) Let us change the definition of $h(x)$ slightly, introducing the factor $1/2\pi$ into the right-hand side of (1) (similarly as in § 2.11). We obtain, then, that $\mathfrak{A}_{\frac{1}{1-\gamma}}[h] \leqslant \mathfrak{A}_{\frac{1}{1-\alpha}}[f]\,\mathfrak{A}_{\frac{1}{1-\beta}}[g]$.

4.17. A theorem of Hardy. *Let* $r > 1$, $s < r - 1$, $f(x) \geqslant 0$, $0 \leqslant x < \infty$, $F(x) = \int_0^x f\,dt$. *If* $f^r(x)\,x^s$ *is integrable over* $(0, \infty)$, *so is* $\{F(x)/x\}^r x^s$, *and*

$$(1) \qquad \int_0^\infty \left\{\frac{F(x)}{x}\right\}^r x^s\,dx \leqslant \left(\frac{r}{r-s-1}\right)^r \int_0^\infty f^r(x)\,x^s\,dx\ ^{1)}.$$

Since $\int_0^x f\,t^{s/r}\,t^{-s/r}\,dt \leqslant \left(\int_0^x f^r\,t^s\,dt\right)^{1/r} \left(\int_0^x t^{-s/(r-1)}\,dt\right)^{(r-1)/r}$, we see that f is integrable over any finite interval and that $F(x) = o(x^{(r-1-s)/r})$ as $x \to 0$. Applying a similar argument to the integral defining $F(x) - F(\xi)$ we obtain that $F(x) - F(\xi) < \frac{1}{2}\varepsilon\,x^{(r-1-s)/r}$, if $x > \xi$ and $\xi = \xi(\varepsilon)$ is large enough. Hence $F(x) = [F(x) - F(\xi)] + F(\xi) < \frac{1}{2}\varepsilon x^{(r-1-s)/r} + O(1) < \varepsilon x^{(r-1-s)/r}$ for x large, and, since $\varepsilon > 0$ is arbitrary, $F(x) = o(x^{(r-1-s)/r})$ as $x \to \infty$.

Let $0 < a < b < \infty$. Integrating by parts, writing $F^{r-1} f x^{s-r+1} = f x^{r/s} \cdot F^{r-1} x^{s-r+1-s/r}$, and applying Hölder's inequality, we obtain

$$\int_a^b \left\{\frac{F}{x}\right\}^r x^s\,dx \leqslant -\left[\frac{F^r x^{s-r+1}}{r-s-1}\right]_a^b + \frac{r}{r-s-1}\left\{\int_a^b f^r x^s\,dx\right\}^{1/r} \left\{\int_a^b \left(\frac{F}{x}\right)^r x^s\,dx\right\}^{1/r'}$$

Dividing both sides by the last factor on the right, and making $a \to 0$, $b \to \infty$, we obtain (1).

4.2. Mean convergence. Let $f_1(x), f_2(x), \ldots$ be a sequence of functions belonging to a class $L^r(a, b)$, $r > 0$. If there exists a function $f(x) \in L^r(a, b)$ such that $\mathfrak{M}_r[f - f_n;\ a, b] \to 0$ as $n \to \infty$, we say that $\{f_n(x)\}$ converges in mean, to $f(x)$, with index r. The following theorem is of fundamental importance.

---

1) See Hardy, Littlewood, and Pólya, *Inequalities*, Chapter IX, where various extensions of this theorem are given.

A necessary and sufficient condition that $\{f_n(x)\}$, $f_n \in L^r(a, b)$, $r \geqslant 1$, *should converge in mean, with index r, to a function* $f(x) \in L^r(a, b)$, *is that* $\mathfrak{M}_r[f_m - f_n]$ *should tend to* 0 *as m and n tend to infinity*[1]).

The necessity of the condition is obvious, since, by Minkowski's inequality, the relations $\mathfrak{M}_r[f - f_m] \to 0$ and $\mathfrak{M}_r[f - f_n] \to 0$ involve $\mathfrak{M}_r[f_m - f_n] \leqslant \mathfrak{M}_r[f - f_m] + \mathfrak{M}_r[f - f_n] \to 0$.

The following remark will be useful in the proof of the sufficiency of the condition.

(i) *If* $\{u_n(x)\}$, $a \leqslant x \leqslant b$, *is a sequence of non-negative functions, and if* $I_1 + I_2 + \ldots < \infty$, *where* I_n *denotes the integral of* u_n *over* (a, b), *then* $u_1(x) + u_2(x) + \ldots$ *converges almost everywhere to a finite function.*

In fact, if the series diverged to $+\infty$ in a set of positive mesure, then, by Lebesgue's theorem on the integration of monotonic sequences, we should have $I_1 + I_2 + \ldots = \infty$.

We will now prove that

(ii) *If* $\mathfrak{M}_r[f_m - f_n] \to 0$ *as* $m, n \to \infty$, *we can find a subsequence* $\{f_{n_k}\}$ *of* $\{f_n\}$ *which converges almost everywhere to a finite function* $f(x)$.

Let $\varepsilon_i = \text{Max}\, \mathfrak{M}_r[f_m - f_n]$ for $m \geqslant i$, $n \geqslant i$. Since $\varepsilon_i \to 0$, we have $\varepsilon_{n_1} + \varepsilon_{n_2} + \ldots < \infty$ if $\{n_k\}$ increases sufficiently rapidly. By Hölder's inequality,

$$(1) \quad \int_a^b |f_{n_k} - f_{n_{k+1}}|\, dx \leqslant (b-a)^{1/r'}\, \mathfrak{M}_r[f_{n_k} - f_{n_{k+1}}] \leqslant \varepsilon_{n_k}(b-a)^{1/r'},$$

and so, in virtue of (i), the series $|f_{n_1}| + |f_{n_2} - f_{n_1}| + |f_{n_3} - f_{n_2}| + \ldots$ converges almost everywhere. The function $f(x) = f_{n_1} + (f_{n_2} - f_{n_1}) + \ldots = \lim f_{n_k}(x)$ exists almost everywhere.

Returning to the proof of the theorem, we observe that, if $n_k > m$, then $\mathfrak{M}_r[f_m - f_{n_k}] \leqslant \varepsilon_m$. Applying Fatou's well-known lemma[2]),

---

[1]) Fischer [1], F. Riesz [1], [2].

[2]) Fatou's lemma may be formulated as follows: if $g_k(x) \geqslant 0$, $k = 1, 2, \ldots$, and $g_k(x) \to g(x)$ almost everywhere in (a, b), then $\int_a^b g_k\, dx \leqslant A$, $k = 1, 2, \ldots$, involves $\int_a^b g\, dx \leqslant A$. In particular, $g(x)$ is integrable over (a, b). See e. g. Saks, *Théorie de l'intégrale*, p. 84.

we obtain that $\mathfrak{M}_r[f_m - f] \leqslant \varepsilon_m$. Thence we conclude that $f \in L^r$ and that $\mathfrak{M}_r[f - f_m] \to 0$ as $m \to \infty$. This completes the proof. We add a few remarks.

(a) In the proof we tacitly assumed that $b - a < \infty$, but the argument holds even when $b - a = \infty$, since (1) subsists if (a, b) is replaced by any finite subinterval (α, β) of (a, b).

(b) The function $f(x)$, the existence of which asserts the theorem, is determined uniquely. In fact, if $\mathfrak{M}_r[f - f_n] \to 0$ and $\mathfrak{M}_r[g - f_n] \to 0$ as $n \to \infty$, then, by Minkowski's inequality, $\mathfrak{M}_r[f-g] \leqslant \mathfrak{M}_r[f-f_n] + \mathfrak{M}_r[f_n - g] \to 0$, i. e. $\mathfrak{M}_r[f-g] = 0$, $f(x) = g(x)$.

(c) We proved the theorem for the case $r \geqslant 1$ because this case is the most interesting in applications, but the result holds also for $0 < r < 1$. In the proof we use, instead of Minkowski's inequality, the first inequality in 4.13(3). In particular, to establish the existence of $f(x)$, we observe that $\{|f_{n_1}| + |f_{n_2} - f_{n_1}| + \dots\}^r \leqslant$ $\leqslant |f_{n_1}|^r + |f_{n_2} - f_{n_1}|^r + \dots$, and that, if we integrate the right-hand side of this inequality over (a, b), we obtain a convergent series, provided that $\varepsilon_{n_1}^r + \varepsilon_{n_2}^r + \dots < \infty$.

4.21. The Riesz-Fischer theorem. Let $\{\varphi_n(x)\}$ be a system of functions, orthogonal and normal in (a, b). We saw in § 1.61 that, if c_n are the Fourier coefficients of a function $f \in L^2$, with respect to $\{\varphi_n\}$, the series $c_0^2 + c_1^2 + \dots$ converges. The converse theorem, due to Riesz and Fischer, is one of the most important achievements of the Lebesgue theory of integration.

Let $\varphi_0, \varphi_1, \varphi_2, \dots$ be an arbitrary system of functions, orthogonal and normal in (a, b), and let $c_0, c_1, c_2, \dots$ be an arbitrary sequence of numbers such that $c_0^2 + c_1^2 + c_2^2 + \dots < \infty$. Then there exists a function $f \in L^2(a, b)$ such that the Fourier coefficient of f with respect to φ_n is c_n, $n = 0, 1, 2, \dots$, and, moreover,

$$(1) \qquad \int_a^b f^2\,dx = \sum_{n=0}^{\infty} c_n^2, \qquad \int_a^b (f - s_n)^2\,dx \to 0 \quad \text{as } n \to \infty,$$

where s_n denotes the n-th partial sum of the series $c_0\varphi_0 + c_1\varphi_1 + \dots$ [1].

[1]) Fischer [1], F. Riesz [1]; see also W. H. and G. C. Young [1], where several alternative proofs are given, and Kaczmarz [2].

From the equation

$$\int_a^b (s_{n+k} - s_n)^2\,dx = \sum_{j=n+1}^{n+k} c_j^2$$

we see that $\mathfrak{M}_2[s_m - s_n] \to 0$ as $m, n \to \infty$. In virtue of the last theorem, there is a function $f \in L^2$ such that $\mathfrak{M}_2[f - s_n] \to 0$ as $n \to \infty$. If $n > j$, we have

(2) $$c_j = \int_a^b s_n \varphi_j\,dx = \int_a^b f\varphi_j\,dx + \int_a^b (s_n - f)\,\varphi_j\,dx.$$

By Hölder's inequality, the last term on the right does not exceed $\mathfrak{M}_2[s_n - f]\,\mathfrak{M}_2[\varphi_j] = \mathfrak{M}_2[s_n - f]$ in absolute value. Hence, making $n \to \infty$, we conclude from (2) that c_j is the Fourier coefficient of f with respect to φ_j, and it remains only to prove the first equation in (1).

In virtue of 4.2(ii), there exists a sequence $\{s_{n_k}(x)\}$ converging to $f(x)$ almost everywhere. Since $\mathfrak{M}_2^2[s_{n_k}] = c_0^2 + c_1^2 + \ldots + c_{n_k}^2 \leqslant c_0^2 + c_1^2 + c_2^2 + \ldots$, an application of Fatou's lemma gives $\mathfrak{M}_2^2[f] \leqslant c_0^2 + c_1^2 + c_2^2 + \ldots$, and this, together with Bessel's inequality $c_0^2 + c_1^2 + c_2^2 + \ldots \leqslant \mathfrak{M}_2^2[f]$, yields the desired result.

4.22. Corollaries. (i) A system $\{\varphi_n(x)\}$, orthogonal and normal in an interval (a, b), is said to be *closed* in this interval if, for any function $f \in L^2(a, b)$, we have the Parseval relation

(1) $$\int_a^b f^2\,dx = \sum_{n=0}^{\infty} c_n^2,$$

where $c_0, c_1, \ldots$ are the Fourier coefficients of f with respect to $\{\varphi_n\}$. *In the domain of functions of the class L^2 the notions of a closed and of a complete system are equivalent.* That every closed system is complete, is obvious. To prove the converse assertion let $c_0, c_1, \ldots$ be the Fourier coefficients of a function $f \in L^2$. Since $c_0^2 + c_1^2 + \ldots < \infty$, there is, by the Riesz-Fischer theorem, a $g \in L^2$ with Fourier coefficients c_n, and such that $\mathfrak{M}_2^2[g] = c_0^2 + c_1^2 + \ldots$ Since f and g have the same Fourier coefficients, and $\{\varphi_n\}$ is complete, we have $f \equiv g$, and the equation (1) follows.

(ii) We know that the trigonometrical system is complete (§ 1.5). Therefore, if a_m, b_m denote the Fourier coefficients of a function $f \in L^2$, and c_m the complex coefficients of f, we have the Parseval equations

(2a) $\frac{1}{\pi}\int_0^{2\pi} f^2\,dx = \frac{1}{2}a_0^2 + \sum_{k=1}^{\infty}(a_k^2+b_k^2)$, (2b) $\frac{1}{2\pi}\int_0^{2\pi} f^2\,dx = \sum_{m=-\infty}^{+\infty}|c_m|^2$

which differ only in notation. It may, however, be observed that they can be obtained independently of the Riesz-Fischer theorem. In view of Bessel's inequality, it is only the inverse inequality which demands a proof. Let $\sigma_n(x)$ be the Fejér sums for the function f; σ_n being a trigonometrical polynomial, we have

$$\frac{1}{\pi}\int_0^{2\pi}\sigma_n^2\,dx = \frac{1}{2}a_0^2 + \sum_{k=1}^{n}(a_k^2+b_k^2)\left(1-\frac{k}{n+1}\right)^2 \leqslant \frac{1}{2}a_0^2 + \sum_{k=1}^{\infty}(a_k^2+b_k^2),$$

and, since $\sigma_n(x)\to f(x)$ almost everywhere, it is sufficient to apply Fatou's lemma.

If we substitute $|f|^2$ for f^2 in the formula (2b), we obtain a formula which holds also for f complex. To show this, let $f=f_1+if_2$, and let c_n, c_n', c_n'' be the complex Fourier coefficients of f, f_1, f_2. If $2c_n' = a_n' - ib_n'$, $2c_n'' = a_n'' - ib_n''$, then $|f|^2 = |f_1|^2 + |f_2|^2$, $c_n = c_n' + ic_n''$, $|c_n|^2 = |c_n'|^2 + |c_n''|^2 + 2(a_n' b_n'' - a_n'' b_n')$. Since the last term on the right is an odd function of n, we obtain that

$$\sum_{n=-\infty}^{+\infty}|c_n|^2 = \sum_{n=-\infty}^{+\infty}(|c_n'|^2+|c_n''|^2) = \frac{1}{2\pi}\int_0^{2\pi}(|f_1|^2+|f_2|^2)\,dx = \frac{1}{2\pi}\int_0^{2\pi}|f|^2\,dx.$$

(iii) *If $f(x)$ is periodic and belongs to $L^2(0, 2\pi)$, the function $\bar f(x)$ defined by the formula*

$$(3)\quad \bar f(x) = -\frac{1}{\pi}\int_0^{\pi}\frac{f(x+t)-f(x-t)}{2\,\mathrm{tg}\,\frac{1}{2}t}\,dt = \lim_{h\to 0}\left\{-\frac{1}{\pi}\int_h^{\pi}\frac{f(x+t)-f(x-t)}{2\,\mathrm{tg}\,\frac{1}{2}t}\,dt\right\}$$

exists almost everywhere and belongs to L^2 [1]. *Moreover* $\bar{\mathfrak{S}}[f] = \mathfrak{S}[\bar f]$.

That $\bar{\mathfrak{S}}[f]$ is the Fourier series of a function $g \in L^2$ follows from Bessel's inequality and the Riesz-Fischer theorem. Consequently, the first arithmetic means $\bar\sigma_n(x; f)$ of $\bar{\mathfrak{S}}[f]$ converge almost everywhere. Thence follows the existence of $\bar f(x)$ (§ 3.32), and since, at almost every point, $\bar\sigma_n(x, f)\to g(x)$, $\bar\sigma_n(x; f)\to \bar f(x)$, we obtain that $g = \bar f$. This completes the proof. We may add that, by Parseval's relation,

$$(4)\qquad \frac{1}{\pi}\int_0^{2\pi} f^2\,dx = \frac{1}{2}a_0^2 + \frac{1}{\pi}\int_0^{2\pi}\bar f^2\,dx.$$

---

[1]) Lusin [1].

4.23. The result (iii) obtained in the preceding section will be generalized in Chapter VII, where it will be shown that the integral 4.22(3) exists almost everywhere for any integrable f. Here we will make a few remarks of a different character.

The existence of $\bar{f}(x)$ is not trivial even when $f(x)$ is continuous. The convergence of this integral is due not to the smallness of $f(x+t)-f(x-t)$ for small t, but to the interference of positive and negative values, for, as we will show, *there exist continuous functions f such that the integral*

(1)
$$\int_0^{\pi} \frac{|f(x+t)-f(x-t)|}{t}\,dt$$

diverges at every point[1]. It will slightly simplify the notation if we consider functions f of period 1 and replace the upper limit of integration π by 1 in the integral (1). We begin by proving the following lemma.

Let $g(x)$, where $|g(x)|\leqslant 1$, $|g'(x)|\leqslant 1$, be a function of period 1, and such that for no value of x the difference $g(x+u)-g(x-u)$ vanishes identically in u[2]*. Then, for $n=2,3,\ldots$, we have*

$$\int_{1/n}^{1} \frac{|g(nx+nt)-g(nx-nt)|}{t}\,dt \geqslant C\log n, \quad \int_0^1 \frac{|g(nx+nt)-g(nx-nt)|}{t}\,dt < C_1\log n,$$

where the constants C and C_1 are independent of n.

Let $nx=y$, $nt=u$. In virtue of the periodicity of g, the first integral takes the form

$$\int_0^1 |g(y+u)-g(y-u)|\left(\frac{1}{u+1}+\ldots+\frac{1}{u+n-1}\right)du \geqslant$$
$$\geqslant\left(\frac{1}{2}+\ldots+\frac{1}{n}\right)\int_0^1 |g(y+u)-g(y-u)|\,du.$$

The first factor on the right exceeds a multiple of $\log n$ and the second, as a continuous, periodic, and non-vanishing function of y, is bounded from below by a positive number. This gives the first part of the lemma. Similarly we obtain the second part, observing that the integral of $|g(y+u)-g(y-u)|/u$ over $(0, 1)$ does not exceed 2.

Let us now put

(2)
$$f(x)=\sum_{n=1}^{\infty} a_n g(\lambda_n x),$$

where the coefficients $a_n>0$ and the integers $\lambda_1<\lambda_2<\ldots$ will be defined in a moment. The integral of $|f(x+t)-f(x-t)|/t$ over $(1/\lambda_\nu, 1)$ is not less than

---

[1] For the divergence almost everywhere of this integral, and of the integrals (4) below, see Lusin [1], 182, Titchmarsh [2], Hardy and Littlewood [4]. For the general result see Kaczmarz [3], [4].

[2] For example, we may take for the curve $y=g(x)$, $0\leqslant x\leqslant 1$ the broken line passing trough the points $(0,0)$, $(^1/_3, {}^1/_3)$, $(1,0)$.

$$a_\nu \int_{1/\lambda_\nu}^{1} \frac{|g(\lambda_\nu x+\lambda_\nu t)-g(\lambda_\nu x-\lambda_\nu t)|}{t}\,dt -$$

$$(3)\qquad -\left(\sum_{n=1}^{\nu-1}+\sum_{n=\nu+1}^{\infty}\right) a_n \int_{1/\lambda_\nu}^{1} \frac{|g(\lambda_n x+\lambda_n t)-g(\lambda_n x-\lambda_n t)|}{t}\,dt \geq C a_\nu \log\lambda_\nu -$$

$$-C_1 \sum_{n=1}^{\nu-1} a_n \log\lambda_n - 2\log\lambda_\nu \sum_{n=\nu+1}^{\infty} a_n,$$

since $|g(\lambda_n x+\lambda_n t)-g(\lambda_n x-\lambda_n t)| \leqslant 2$. If we put $a_n = 1/n!$, $\lambda_n = 2^{(n!)^2}$ the right-hand side of (3) divided by $\nu!$ tends to $C\log 2 > 0$, and this proves that (1) diverges everywhere.

It is interesting to observe that the integrals

$$(4)\qquad \int_0^\pi \frac{f(x+t)-f(x)}{t}\,dt, \qquad \int_0^\pi \frac{f(x+t)+f(x-t)-2f(x)}{t}\,dt,$$

apparently similar to the integral 4.22(3), may diverge everywhere for f continuous. The proof, although analogous to that given above, is slightly less simple. See also § 3.9.5.

4.3. We have proved that the necessary and sufficient condition that numbers $a_0, a_1, b_1, \ldots$ should be the Fourier coefficients of a function $f \in L^2$ is that $\frac{1}{2}a_0^2 + (a_1^2+b_1^2) + \ldots$ should converge. The question arises if anything so simple can be proved for the classes L^r with $r \neq 2$. The answer is negative and it is just this answer which makes the Riesz-Fischer theorem and the Parseval relation such an exceptional tool of investigation. Postponing to a later chapter the discussion of some partial results which may be obtained in this direction, we will consider here criteria of a different kind, involving the Cesàro or Abel means of the series considered.

Besides the classes L_φ, L^r introduced in § 4.1 we shall consider the classes B of bounded and C of continuous, periodic functions. If a trigonometrical series

$$(1)\qquad \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx) = \sum_{n=-\infty}^{+\infty} c_n e^{inx}$$

is a $\mathfrak{S}[f]$, with f belonging to L_φ, B or C, we shall say that the series (1) itself belongs to L_φ, B, C respectively. By S we shall denote the class of Fourier-Stieltjes series.

The first arithmetic means of the series (1) will be denoted by $\sigma_n(x)$.

4.31. Classes B **and** C. *A necessary and sufficient condition that the series* 4.3(1) *should belong to C is the uniform convergence of the sequence* $\{\sigma_n(x)\}$. The necessity is nothing else but Fejér's theorem. To prove the sufficiency, we observe that, for $n > |k|$, we have

$$(2) \qquad \left(1 - \frac{|k|}{n+1}\right) c_k = \frac{1}{2\pi} \int_0^{2\pi} \sigma_n(x) e^{-ikx} dx.$$

As $n \to \infty$, the left-hand side tends to c_k, and the expression on the right to the Fourier coefficient of the function $f(x) = \lim \sigma_n(x)$.

A necessary and sufficient condition that 4.3(1) *should belong to B, is the existence of a constant K such that* $|\sigma_n(x)| \leqslant K$ *for all x and n.* The necessity was proved in § 3.22, with K equal to the essential upper bound of $|f|$. Conversely, if $|\sigma_n| \leqslant K$, we obtain that

$$2K^2 \geqslant \frac{1}{\pi} \int_0^{2\pi} \sigma_n^2 dx = \tfrac{1}{2} a_0^2 + \sum_{n=1}^{n} (a_k^2 + b_k^2)\left(1 - \frac{k}{n+1}\right)^2 \geqslant$$

$$\geqslant \tfrac{1}{2} a_0^2 + \sum_{k=1}^{\nu} (a_k^2 + b_k^2)\left(1 - \frac{k}{n+1}\right)^2,$$

where $\nu > 0$ is any fixed integer less than n. Making $n \to \infty$ we see that $\frac{1}{2} a_0^2 + (a_1^2 + b_1^2) + \dots + (a_\nu^2 + b_\nu^2) \leqslant 2K^2$. Since ν is arbitrary, the series $\frac{1}{2} a_0^2 + (a_1^2 + b_1^2) + \dots$ converges, and so 4.3(1) is a $\mathfrak{S}[f]$ with $f \in L^2$. Therefore $\sigma_n(x) \to f(x)$ almost everywhere, and the inequalities $|\sigma_n(x)| \leqslant K$, imply that $|f(x)| \leqslant K$ almost everywhere.

4.32. The class S. *A necessary and sufficient condition that the series* 4.3(1) *should belong to S is that* $\mathfrak{M}[\sigma_n] \leqslant V$, *where V is a finite constant independent of n* [1]).

If 4.3(1) is a $\mathfrak{S}[dF]$, then

$$(1) \quad \sigma_n(x) = \frac{1}{\pi} \int_0^{2\pi} K_n(x-t)\, dF(t), \quad |\sigma_n(x)| \leqslant \frac{1}{\pi} \int_0^{2\pi} K_n(x-t)\, |dF(t)| \text{ [2]}).$$

Integrating this inequality with respect to x, and interchanging the order of integration on the right [3]), we find that

---

[1]) Young [8].

[2]) $|dF(t)|$ means the same as $dV(t)$, where $V(t)$ is the total variation of F over $(0, t)$.

[3]) Since $K_n(u)$ is continuous, the justification of this procedure is immediate: we may replace the integral of $|\sigma_n(x)|$ by approximate Riemannian sums and interchange the order of summation and integration.

$$\mathfrak{M}[\sigma_n] \leqslant \frac{1}{\pi}\int_0^{2\pi} |dF(t)| \int_0^{2\pi} K_n(x-t)\,dx = \int_0^{2\pi} |dF(t)| = V,$$

where V is the total variation of F over $(0, 2\pi)$. For the second part of the theorem we need the following important lemma.

4.321. *Given a sequence of functions $\{F_n(x)\}$, $a \leqslant x \leqslant b$, of uniformly bounded variation, either there exists a uniformly bounded subsequence $\{F_{n_k}(x)\}$ converging everywhere to a function $F(x)$ of bounded variation, or $\{|F_n(x)|\}$ diverges uniformly to $+\infty$ as $n \to \infty$* [1].

Suppose first that all the functions F_n are non-negative, non-decreasing and less than a constant V. Let $R = \{r_n\}$ be the sequence consisting of all the rational points from (a, b) and of the points a, b. $\{F_n(r_1)\}$ being bounded, we can find a sequence $(S_1)\ p_1^1, p_2^1, \dots p_k^1, \dots$ of indices, such that $\{F_{p_k^1}(r_1)\}$ converges. Rejecting the first term p_1^1, we find from the remaining indices $p_2^1, p_3^1, \dots$ a subsequence $(S_2)\ p_1^2, p_2^2, \dots p_k^2, \dots$ such that $\{F_{p_k^2}(r_2)\}$ converges. Rejecting p_1^2, we choose among the rest a subsequence $(S_3)\ p_1^3, p_2^3, \dots$ such that $\{F_{p_k^3}(r_3)\}$ converges and so on. The sequence $p_1^1, p_1^2, p_1^3, \dots$ being, from some place onwards, a subsequence of every S_i, we see that $\{F_{p_1^k}(x)\}$ converges, at least for rational x, to a limit $F(x)$, non-decreasing over the set where it exists.

For any x interior to (a, b) put $d(x) = \lim_{r\to x+0} F(r) - \lim_{r\to x-0} F(r)$, $r \in R$. Since for any system $x_1, x_2, \dots, x_n$ we have $d(x_1) + \dots + d(x_n) \leqslant V$, it follows that the number of the points x where $d(x) \geqslant \varepsilon > 0$ is finite. Let Z be the at most enumerable set of points for which $d(x) > 0$. We will prove that, for any $x \bar{\in} Z$, $\lim F_{p_1^k}(x)$ exists. In fact, given an arbitrary $\eta > 0$ and an $x \bar{\in} Z$, $x \neq a, b$, we can find two rational points $r' < x < r''$, such that $0 \leqslant F(r'') - F(r') < \eta$. Since $F_{p_1^k}(r') \leqslant F_{p_1^k}(x) \leqslant F_{p_1^k}(r'')$, where the extreme terms tend to $F(r'), F(r'')$ as $k \to \infty$, we see that the oscillation of $\{F_{p_1^k}(x)\}$ does not exceed η, i. e. the sequence converges.

Let D be the set of points where $\{F_{p_1^k}(x)\}$ diverges; D is at most enumerable. Repeating with D the same argument as with R, we

---

[1]) Helly [1].

find a subsequence $\{n_k\}$ of $\{p_1^k\}$ such that $\{F_{n_k}(x)\}$ converges in D, i. e. everywhere in (a, b).

In the general case we put $F_n(x) = F_n(a) + P_n(x) - N_n(x)$, where $P_n(x)$ and $N_n(x)$ denote the positive and negative variations of $F_n(x) - F_n(a)$. Let us suppose that we can find a sequence $\{m_k\}$ such that $\{F_{m_k}(a)\}$ converges to a finite limit. From $\{m_k\}$ we choose a subsequence $\{m'_k\}$ such that $\{P_{m'_k}(x)\}$ converges, and from $\{m'_k\}$ a subsequence $\{n_k\}$ such that $\{N_{n_k}(x)\}$, and therefore $\{F_{n_k}(x)\}$ converges. That $F(x) = \lim F_{n_k}(x)$ is of bounded variation, follows from the fact that $F(x) = \lim F_{n_k}(a) + \lim P_{n_k}(x) - \lim N_{n_k}(x)$, where the last two terms are non-decreasing and bounded functions of x.

If our assumption concerning $\{F_n(a)\}$ does not hold, then $|F_n(a)| \to \infty$. Since the oscillations of the functions $F_n(x)$ are uniformly bounded, it is easy to see that $\{|F_n(x)|\}$ diverges uniformly to $+\infty$ as $n \to \infty$. This completes the proof of the lemma.

The following remark will be useful later. If the total variations $P_n(b) + N_n(b)$ of the functions F_n do not exceed a number W, the same is true for the total variation of F.

4.322. Suppose now, in the case of Theorem 4.32, the condition $\mathfrak{M}[\sigma_n] \leqslant V$ satisfied. Let $F_n(x)$ be the integral of $\sigma_n(t)$ over $(0, x)$. The functions $F_n(x)$ are of uniformly bounded variation over $(0, 2\pi)$. Since $F_n(0) = 0$, $n = 1, 2, \dots$, $\{|F_n(x)|\}$ cannot diverge to $+\infty$ and so there exists a sequence $\{F_{n_j}(x)\}$ uniformly bounded and converging everywhere to a function $F(x)$ of bounded variation. Let $n_j > |k|$. Integrating by parts, and making $j \to \infty$, we obtain

$$\left(1 - \frac{|k|}{n_j + 1}\right) c_k = \frac{1}{2\pi}\int_0^{2\pi} \sigma_{n_j} e^{-ikx}\, dx = \frac{1}{2\pi} F_{n_j}(2\pi) + \frac{ik}{2\pi}\int_0^{2\pi} F_{n_j} e^{-ikx}\, dx,$$

$$c_k = \frac{1}{2\pi} F(2\pi) + \frac{ik}{2\pi}\int_0^{2\pi} F e^{-ikx}\, dx = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikx}\, dF(x),$$

for $k = 0, \pm 1, \dots$, so that 4.3(1) is $\mathfrak{S}[dF]$. We complete the theorem by a few remarks.

4.323. *If* 4.3(1) *is a* $\mathfrak{S}[dF]$, *where* $F(x) = \frac{1}{2}[F(x+0) + F(x-0)]$ *for every* x *and if the total variation of* F *over* $(0, 2\pi)$ *is* V, *then* $\mathfrak{M}[\sigma_n] \to V$ *as* $n \to \infty$. It has been proved in § 4.32 that $\overline{\lim}\, \mathfrak{M}[\sigma_n] \leqslant V$, and it remains only to show that the assumption $\underline{\lim}\, \mathfrak{M}[\sigma_n] < W < V$ leads to a contradiction.

In fact, let $\{m_j\}$ be such that $\mathfrak{M}[\sigma_{m_j}] \leqslant W$. The sequence $\{F_{n_j}\}$ considered in the preceding section may, plainly, be chosen from $\{F_{m_j}\}$ and, by the final remark of § 4.321, the total variation of $F_*(x) = \lim F_{n_j}(x)$ would not exceed W. Without loss of generality we may assume that $F_*(x) = \frac{1}{2}[F_*(x+0) + F_*(x-0)]$, for if we replace $F_*(x)$ by $\frac{1}{2}[F_*(x+0) + F_*(x-0)]$ at every point of discontinuity, the total variation of the function will not increase. Since $\mathfrak{S}[dF]$ and $\mathfrak{S}[dF_*]$ have the same coefficients, it follows that the difference $F_1(x) = F(x) - F_*(x)$ is equal to a constant C at almost every point x. On the other hand we have $F_1(x) = \frac{1}{2}[F_1(x+0) + F_1(x-0)]$, so that $F_1(x) = C$ for every x. Hence the total variations of F and F_* over $(0, 2\pi)$ are equal, contrary to what we assumed.

4.324. *A necessary and sufficient condition that* 4.3(1) *should be a* $\mathfrak{S}[dF]$ *with* F *non-decreasing is* $\sigma_n(x) \geqslant 0$, $n = 0, 1, 2, \ldots$

The necessity follows from the first formula 4.32(1) since $K_n \geqslant 0$. Conversely, if $\sigma_n(x) \geqslant 0$, the functions $F_n(x)$ considered in § 4.322 are non-decreasing, and the same is true for $F(x) = \lim F_{n_j}(x)$.

4.325. *A necessary and sufficient condition that* 4.3(1) *should be the Fourier series of a function of bounded variation is that* $\mathfrak{M}[\sigma'_n] = O(1)$. This theorem is equivalent to Theorem 4.32 (§ 2.14).

4.326. Carathéodory's theorem. Let $\{F_k(x)\}$, $0 \leqslant x < 2\pi$, be a uniformly bounded sequence of functions. If $F_k(x)$ tends almost everywhere to a limit $F(x)$, then $c_n^k \to c_n$ as $k \to \infty$, where c_n^k, c_n, $n = 0, \pm 1, \ldots$ denote the Fourier coefficients of the functions $F_k(x)$, $F(x)$ respectively. Simple examples show that, without additional conditions, the converse theorem is false, and it is an important fact that this converse theorem is true when the functions $F_k(x)$ are monotonic. More precisely:

Let $\{F_k(x)\}$, $0 \leqslant x < 2\pi$ *be a sequence of uniformly bounded and non-decreasing functions, and let* c_n^k *be the complex Fourier coefficients of* F_k. *If, for* $n = 0, \pm 1, \pm 2, \ldots$, *we have* $\lim c_n^k = c_n$ *as* $k \to \infty$, *the numbers* c_n *are the Fourier coefficients of a monotonic function* $F(x)$, *and* $F_k(x) \to F(x)$ *at every point* x, $0 < x < 2\pi$, *where* $F(x)$ *is continuous*[1].

[1] Carathéodory [2].

In virtue of Theorem 4.321 there is a subsequence $\{F_{k_j}\}$ of $\{F_k\}$ converging everywhere to a non-decreasing function $F(x)$. It is plain that the Fourier coefficients of F are c_n, and we have only to show that $F_n(x) \to F(x)$ except, perhaps, at the set of points where F is discontinuous. Let ξ, $0 < \xi < 2\pi$, be a point of continuity of $F(x)$. Let us suppose that $F_k(\xi)$ does not tend to $F(\xi)$. We can then find a sequence $\{F_{k_i}\}$ such that $\lim F_{k_i}(\xi)$ exists and is $\neq F(\xi)$. To fix ideas let us suppose that $\lim F_{k_i}(\xi) > F(\xi)$. We can find a subsequence $\{F_{l_i}(x)\}$ of $\{F_{k_i}(x)\}$ such that $\lim F_{l_i}(x) = G(x)$ exists everywhere. The Fourier coefficients of G are c_n, and so $F(x) \equiv G(x)$. On the other hand $G(\xi) = \lim F_{l_i}(\xi) = \lim F_{k_i}(\xi) > F(\xi)$, and, since $G(x)$ is non-decreasing and $F(x)$ is continuous for $x = \xi$, we have $G(x) > F(x)$ in an interval $\xi \leqslant x \leqslant \xi + h$, $h > 0$, so that $G(x) \not\equiv F(x)$. This contradiction shows that $F_k(\xi) \to F(\xi)$.

4.33. Classes L_φ [1]). *Let $\varphi(u)$, $u \geqslant 0$, be convex, non-negative, and such that $\varphi(u)/u \to \infty$ as $u \to \infty$* [2]). *A necessary and sufficient condition that* 4.3(1) *should belong to L_φ is that $\mathfrak{M}[\varphi|\sigma_n|] \leqslant C$, where C is finite and independent of n* [3]).

We may suppose that $\varphi(u)$ is non-decreasing, for otherwise it is sufficient to consider the function $\varphi^*(u)$ equal to $\varphi(u)$ for $u \geqslant u_0$ and to $\varphi(u_0)$ for $0 \leqslant u \leqslant u_0$, u_0 denoting the point where φ attains its minimum. The classes L_φ and L_{φ^*} are plainly identical.

To prove the necessity of the condition consider the inequality

$$(1) \qquad |\sigma_n(x)| \leqslant \frac{1}{\pi}\int_0^{2\pi} K_n(x-t)\,|f(t)|\,dt.$$

By Jensen's theorem, and taking into account that the integral of the function $p(t) = K_n(x-t)/\pi$ over $(0, 2\pi)$ is equal to 1, we find that

$$(2) \qquad \varphi|\sigma_n(x)| \leqslant \frac{1}{\pi}\int_0^{2\pi} \varphi|f(t)|\,K_n(x-t)\,dt.$$

Integrating this with respect to x and inverting the order of integration, we find the important inequality

---

[1]) Young [10], see also Zygmund [4].

[2]) It follows that φ is bounded in any finite interval.

[3]) We write $\varphi|\sigma_n|$ instead of $\varphi(|\sigma_n|)$.

(3) $$\mathfrak{M}[\varphi|\sigma_n|] \leqslant \mathfrak{M}[\varphi|f|],$$

which gives the first half of the theorem.

As regards the second half, the Jensen inequality $\varphi(\mathfrak{M}[\sigma_n]/2\pi) \leqslant \leqslant \mathfrak{M}[\varphi|\sigma_n|]/2\pi \leqslant C/2\pi$ implies that $\mathfrak{M}[\sigma_n] = O(1)$, i. e. the series 4.3(1) is a $\mathfrak{S}[dF]$ (§ 4.32). To prove that $F(x)$ is absolutely continuous, it is sufficient to show that the functions $F_n(x) = \int_0^x \sigma_n(t)\,dt$ are uniformly absolutely continuous, i. e. that, given an $\varepsilon > 0$, there exists a $\delta > 0$ such that, for any finite system S of non-overlapping intervals $(a_1, b_1), (a_2, b_2), \dots, (b_1 - a_1) + (b_2 - a_2) + \dots < \delta$, we have

(4) $$\sum_i |F_n(b_i) - F_n(a_i)| \leqslant \varepsilon, \quad n = 1, 2, \dots$$ [1].

The inequality

$$\varphi\left(\frac{1}{|S|}\int_S |\sigma_n(x)|\,dx\right) \leqslant \frac{\int_0^{2\pi} \varphi|\sigma_n|\,dx}{|S|} \leqslant \frac{C}{|S|}$$

may be written in the form $\varphi(\xi u)/\xi u \leqslant C/\xi$, where $u = 1/|S|$, $\xi = \int_S |\sigma_n|\,dx$. In view of our hypothesis concerning φ, we see that if $u \to \infty$, then $\xi \to 0$, and so if $|S|$ is sufficiently small, then $\xi < \varepsilon$.

Since the left-hand side of (4) does not exceed ξ, the absolute continuity of F follows.

Let $F'(x) = f(x)$. The series 4.3(1) is $\mathfrak{S}[f]$. To show that $f \in L_\varphi$, we observe that $\sigma_n \to f$ almost everywhere, and, applying Fatou's lemma to the inequality $\mathfrak{M}[\varphi|\sigma_n|] \leqslant C$, we find that $\mathfrak{M}[\varphi|f|] \leqslant C$.

As a corollary we obtain that *a necessary and sufficient condition that* 4.3(1) *should belong to* L^r, $r > 1$, *is that* $\mathfrak{M}_r[\sigma_n] = O(1)$[2]. As Theorem 4.32 shows, this result does not hold for $r = 1$.

4.34. *A necessary and sufficient condition that* 4.3(1) *should be a Fourier series is that* $\mathfrak{M}[\sigma_m - \sigma_n] \to 0$ *as* $m, n \to \infty$[3].

---

[1]) In fact, if, for fixed S, the inequality (4) is satisfied by the functions F_n, it is also satisfied by $F = \lim F_n$.

[2]) W. H. and G. C. Young [1].

[3]) Steinhaus [2], Gross [1].

Let us suppose that 4.3(1) is a $\mathfrak{S}[f]$. Integrating the inequality

$$(1) \qquad |\sigma_n(x) - f(x)| \leqslant \frac{1}{\pi} \int_{-\pi}^{\pi} |f(x+t) - f(x)| K_n(t)\, dt$$

over $(0, 2\pi)$, we find that

$$(2)\ \mathfrak{M}[\sigma_n - f] \leqslant \frac{1}{\pi} \int_{-\pi}^{\pi} \eta(t) K_n(t)\, dt, \text{ where } \eta(t) = \int_{-\pi}^{\pi} |f(x+t) - f(x)|\, dx.$$

Since $\eta(t)$ is continuous and vanishes for $t = 0$, and the right-hand side of the last inequality is the n-th Fejér sum of $\mathfrak{S}[\eta]$ at $t = 0$, we see that $\mathfrak{M}[\sigma_n - f] \to 0$, and so $\mathfrak{M}[\sigma_m - \sigma_n] \leqslant \mathfrak{M}[\sigma_m - f] + \mathfrak{M}[\sigma_n - f] \to 0$ as $m, n \to \infty$.

Conversely, the condition $\mathfrak{M}[\sigma_m - \sigma_n] \to 0$ implies $\mathfrak{M}[\sigma_n] = O(1)$, i. e. 4.3(1) is a $\mathfrak{S}[dF]$. To show that F is absolutely continuous, it is enough to prove (as in § 4.33) that $\mathfrak{M}[\sigma_n; S]$ [1]) is small with $|S| = (b_1 - a_1) + (b_2 - a_2) + \dots$, uniformly in n. Now $\mathfrak{M}[\sigma_n; S] \leqslant \mathfrak{M}[\sigma_n - \sigma_\nu; S] + \mathfrak{M}[\sigma_\nu; S] \leqslant \mathfrak{M}[\sigma_n - \sigma_\nu; 0, 2\pi] + \mathfrak{M}[\sigma_\nu; S]$. Let ν be so large that $\mathfrak{M}[\sigma_n - \sigma_\nu] < \frac{1}{2}\varepsilon$ for $n > \nu$. For fixed ν we have $\mathfrak{M}[\sigma_\nu; S] < \frac{1}{2}\varepsilon$ if only $|S| < \delta = \delta(\varepsilon)$. Therefore $\mathfrak{M}[\sigma_n; S] < \varepsilon$ for $n > \nu$, $|S| < \delta$, and this completes the proof.

4.35. Suppose that a convex and non-negative function $\varphi(u)$ satisfies the condition $\varphi(0) = 0$, so that φ is non-decreasing. Assuming that 4.3(1) belongs to L_φ, we may ask under what conditions $\mathfrak{M}[\varphi|\sigma_n - f|] \to 0$. Starting from 4.34(1) and using an argument similar to that of § 4.34, we see that $\mathfrak{M}[\varphi|\sigma_n - f|] \to 0$, if only the function

$$(1) \qquad \eta(t) = \int_{-\pi}^{\pi} \varphi\{|f(x+t) - f(x)|\}\, dx.$$

is integrable and tends to 0 with t. This may not be true if φ increases too rapidly, but an insertion of the factor $^1/_4$ into curly brackets saves the situation: if $f \in L_\varphi$, then the function $\mathfrak{M}[\varphi\{^1/_4|f(x+t) - f(x)|\}]$ is integrable and tends to 0 with t. In fact, let $f = g + h$, where g is bounded and $\mathfrak{M}[\varphi|h|] < \varepsilon$. By Jensen's inequality we have

[1]) This symbol denotes the integral of $|\sigma_n|$ over S.

$$\mathfrak{M}\,[\varphi\,\{{}^1\!/_4\,|f(x+t)-f(x)|\}] \leqslant \tfrac{1}{2}\,\mathfrak{M}\,[\varphi\,\{\tfrac{1}{2}\,|g(x+t)-g(x)|\}] + \\ + \tfrac{1}{2}\,\mathfrak{M}\,[\varphi\,\{\tfrac{1}{2}\,|h(x+t)-h(x)|\}].$$

The last term on the right does not exceed ${}^1\!/_4\,\mathfrak{M}\,[\varphi\,|h(x+t)|\,] + {}^1\!/_4\,\mathfrak{M}\,[\varphi\,|h|\,] < \varepsilon/2$, and, since the preceding term tends to 0, [1]) the left-hand side is less than ε for $|t|$ sufficiently small.

At the same time we have proved that, *if the series* 4.3(1) *is a* $\mathfrak{S}[f]$ *with* $f \in L_\varphi$, *where* $\varphi(u)$ *is convex, non-negative, and* $\varphi(0) = 0$, *then*

$$\mathfrak{M}\,[\varphi\,\{{}^1\!/_4\,|f-\sigma_n|\}] \to 0 \quad as \quad n \to \infty.$$

In particular, *if* $f \in L^r$, $r \geqslant 1$, *then* $\mathfrak{M}_r[f-\sigma_n] \to 0$ [2]).

4.36. Abel means. So far we have worked with Fejér's kernel. The essential property of this kernel, viz. positiveness, is shared by some other kernels, in particular by Poisson's kernel. Therefore all our results remain true for Abel's method of summation, which, as we know, has a very important function-theoretic significance. Since the proofs are esentially the same as before [3]), we content ourselves with stating the results [4]). By $f(r, x)$ we mean the harmonic function corresponding to the series 4.3(1).

(i) *A necessary and sufficient condition that* 4.3(1) *should belong to C is that* $f(r, x)$ *should converge uniformly as* $r \to 1$; *a necessary and sufficient condition that* 4.3(1) *should belong to B, is that* $f(r, x)$ *should be bounded for* $0 \leqslant r < 1$, $0 \leqslant x \leqslant 2\pi$.

(ii) *A necessary and sufficient condition that* $f(r, x)$ *should satisfy a relation*

$$f(r, x) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(t-x)+r^2}\,dF(t),$$

where F is of bounded variation, is that $\mathfrak{M}\,[f(r, x)] = O(1)$ *as* $r \to 1$. *If V is the total variation of F over* $(0, 2\pi)$, *and if* $2F(x) = F(x+0) +$

---

[1]) From our hypothesis concerning φ it follows that, in any finite interval $0 \leqslant u \leqslant a$, we have $\varphi(u) \leqslant Mu$, with $M = M(a)$.

[2]) W. H. and G. C. Young [1].

[3]) That in Abel's method the variable changes continuously is quite immaterial, since we may consider any sequence $\{r_n\}$ tending to 1.

[4]) See also: Evans, *The logarithmic potential*, Fichtenholz [1]. F. Riesz [10].

$+F(x-0)$ *for every* x, *then* $\mathfrak{M}[f(r,x)]\to V$ *as* $r\to 1$. *F is non-decreasing if and only if* $f(r,x)\geqslant 0$.

(iii) Let $\varphi(u)$ satisfy the hypothesis of Theorem 4.33. Then *a necessary and sufficient condition that* 4.3(1) *should belong to* L_φ *is that* $\mathfrak{M}[\varphi(|f(r,x)|)]=O(1)$ *as* $r\to 1$.

If 4.3(1) *is a* $\mathfrak{S}[f]$ *with* $f\in L_\varphi$, *then* $\mathfrak{M}[\varphi\{\frac{1}{4}|f(x)-f(r,x)|\}]\to 0$ *as* $r\to 1$. *If* $f\in L^r$, $r\geqslant 1$, *then* $\mathfrak{M}_r[f(x)-f(r,x)]\to 0$.

(iv) *The series* 4.3(1) *is a Fourier series if and only if* $\mathfrak{M}[f(r,x)-f(\rho,x)]\to 0$ *as* $r,\rho\to 1$.

4.37. (C,k) **means.** Most of the results remain true, although some inequalities become less precise, for quasi-positive kernels, in particular for the (C,k) kernels, $k>0$. Let $\lambda_n=\lambda_n^{(k)}$ denote the integral of $|K_n^k(u)|/\pi$ over $(0,2\pi)$, and $\lambda=\lambda^{(k)}$ the upper bound of $\{\lambda_n^{(k)}\}$, $n=1,2,\ldots$ We quote the following theorems, the proofs of which follow immediately.

(i) *If* $\mathfrak{M}[\varphi(|\sigma_n^k|)]=O(1)$, *then the series* 4.3(1) *belongs to* L_φ. *If* 4.3(1) *is a* $\mathfrak{S}[f]$ *with* $f\in L_\varphi$, *then* $\mathfrak{M}[\varphi|\lambda^{-1}\sigma_n^k|]=O(1)$, *and* $\mathfrak{M}[\varphi\{|\sigma_n^k-f|/4\lambda\}]=o(1)$. In particular, *a necessary and sufficient condition that* 4.3(1) *should belong to* L^r, $r>1$, *is that* $\mathfrak{M}_r[\sigma_n^k]=O(1)$. *If* 4.3(1) *is a* $\mathfrak{S}[f]$ *with* $f\in L^r$, $r\geqslant 1$, *then* $\mathfrak{M}_r[f-\sigma_n^k]\to 0$ *as* $n\to\infty$.

(ii) *A necessary and sufficient condition that* 4.3(1) *should belong to S is that* $\mathfrak{M}[\sigma_n^k]=O(1)$.

(iii) *A necessary and sufficient condition that* 4.3(1) *should belong to L is that* $\mathfrak{M}[\sigma_m^k-\sigma_n^k]\to 0$ *as* $m,n\to\infty$.

4.38. Let us replace σ_n by the partial sums s_n in the theorems of §§ 4.31—4.35. The conditions which we obtain remain sufficient (although, as we shall see later, some of them are no longer necessary). The proofs are similar, except at one point: we cannot use the fact that if 4.3(1) is a $\mathfrak{S}[f]$, then $s_n(x)\to f(x)$ almost everywhere, for such a theorem is false. But for our purposes it is sufficient to assume that there exists a *subsequence* $\{s_{n_k}(x)\}$ of $\{s_n(x)\}$ converging to f almost everywhere, and we shall see in § 7.3 that this is certainly true if $\{n_k\}$ increases sufficiently rapidly.

4.39. In the sufficiency-parts of the theorems of §§4.31—4.38 it is enough to assume that the conditions imposed upon $\sigma_n(x)$, $f(r,x)$, or $s_n(x)$, are satisfied not for all indices n, r but only for a sequence of them. The proofs require no changes.

Thus if, for a sequence $n_1 < n_2 < \dots$, $\{s_{n_k}\}$ or $\{\sigma_{n_k}\}$ converges uniformly, the series 4.3(1) belongs to C. If $\mathfrak{M}[s_{n_k}] = O(1)$, the series belongs to S, etc.

This enables us to state some of the theorems given above in a slightly different form. For example, *a necessary and sufficient condition that* 4.3(1) *should belong to C is that the functions $\sigma_n(x)$ should be uniformly continuous.* The necessity follows from the inequality 4.33(1), which, applied to $f(t+h)-f(t)$, shows that $\omega(\delta; \sigma_n) \leqslant \omega(\delta; f)$ (§ 2.2). Conversely, if the functions $\sigma_n(x)$ are uniformly continuous, there exists a sequence $\{\sigma_{n_k}(x)\}$ converging uniformly to a continuous function $f(x)$ [1]), and so the series is $\mathfrak{S}[f]$, $f \in C$.

4.4. Parseval's relations. Let f and g be two functions of the class L^2, with Fourier coefficients a_n, b_n and a'_n, b'_n respectively. Adding the Parseval formulae 4.22(2a) formed for $f+g$ and $f-g$, we obtain

$$(1) \qquad \frac{1}{\pi}\int_0^{2\pi} fg\,dx = \frac{a_0 a'_0}{2} + \sum_{n=1}^{\infty}(a_n a'_n + b_n b'_n),$$

where the series on the right converges absolutely. The formula (1), which is called Parseval's relation for f and g, holds in other cases besides the one in which $f \in L^2$, $g \in L^2$ [2]). Two classes of functions K and K_1 will be called *complementary* classes if (1) holds for every $f \in K$, $g \in K_1$. The series on the right need not be convergent; we shall only suppose that it is summable by some method of summation. It will appear that the Fourier series of functions belonging to complementary classes have, in some cases, much the same, or analogous, properties, and Parseval's formula (1), where f and g enter symmetrically, is just the means to discover these properties in common.

4.41. *The following are pairs of complementary classes:* (i) L_Φ *and* L_Ψ, *where* Φ *and* Ψ *are Young's complementary functions,* (ii) L^r *and* $L^{r'}$ $(r > 1)$, (iii) B *and* L, (iv) C *and* S. *In all these cases the series in* 4.4(1) *is summable* $(C, 1)$.

---

[1]) We apply here Arzela's well-known theorem on families of uniformly continuous functions. See e. g. Hobson, *Theory of functions* 2, 168.

[2]) The formula is obvious if one of the functions f and g is a trigonometrical polynomial. The series on the right consists then of a finite number of terms.

Part (iv) of the theorem is to be understood in the sense that, if a_n, b_n are the coefficients of a $\mathfrak{S}[f]$, $f \in C$, and a'_n, b'_n are the coefficients of a $\mathfrak{S}[dG] \in S$, then we have the formula 4.4(1) with $fg\,dx$ replaced by FdG. Part (iii) is a limiting case ($r = \infty$) of (ii).

Let $\sigma_n(x)$ be the $(C, 1)$ means of $\mathfrak{S}[f]$, τ_n the $(C, 1)$ means of the series in 4.4(1), and Δ_n the difference between the left-hand side of 4.4(1) and τ_n. We have then

$$(1) \qquad \Delta_n = \frac{1}{\pi}\int_0^{2\pi} (f - \sigma_n)\, g\, dx,$$

and, applying Hölder's inequality, we see that $|\Delta_n|$ does not exceed $\pi^{-1}\mathfrak{M}_r[f-\sigma_n]\ \mathfrak{M}_{r'}[g] \to 0$ as $n \to \infty$. This proves part (ii) of the theorem. To establish part (i), which embraces (ii), we apply Young's inequality to $|\Delta_n|/16$:

$$\pi\,|\Delta_n|/16 \leqslant \mathfrak{M}\,[\Phi\,\{^1/_4\,|f-\sigma_n|\}] + \mathfrak{M}\,[\Psi\,\{^1/_4\,|g|\}].$$

From Theorem 4.35, we obtain that $\overline{\lim}\,\Delta_n \leqslant 16\pi^{-1}\,\mathfrak{M}\,[\Psi\{^1/_4\,|g|\}]$. Let $g = g' + g''$, where g' is a trigonometrical polynomial and $\mathfrak{M}\,[\Psi\,\{^1/_4\,|g''|\}] < \varepsilon$ [1]). Substituting, in (1), g' and g'' for g, we obtain expressions Δ'_n and Δ''_n, such that $\Delta_n = \Delta'_n + \Delta''_n$. Since g' is only a polynomial, we see from Parseval's formula for f and g' that $\Delta'_n \to 0$. On the other hand, $\overline{\lim}\,\Delta''_n \leqslant 16\pi^{-1}\,\mathfrak{M}\,[\Psi\,\{^1/_4\,|g''|\}] < 16\varepsilon/\pi$. Since $\overline{\lim}\,\Delta_n \leqslant \overline{\lim}\,\Delta'_n + \overline{\lim}\,\Delta''_n < 16\varepsilon/\pi$, where ε is arbitrary, we infer that $\Delta_n \to 0$.

If f is bounded, $|f| \leqslant M$, g integrable, then $|f-\sigma_n|\,|g|$ tends to 0 almost everywhere and is majorised by the integrable function $2M\,|g|$. Applying Lebesgue's theorem on the integration of sequences, we conclude from (1) that $\Delta_n \to 0$.

Finally, to prove (iv), let us replace in (1) $g(x)$ by $dG(x)$. Since $\pi\,|\Delta_n|$ does not exceed $\mathrm{Max}\,|f(x) - \sigma_n(x)|$, $0 \leqslant x \leqslant 2\pi$, multiplied by the total variation of G over $(0, 2\pi)$, we have again $\Delta_n \to 0$, provided that f is continuous.

4.411. Let $g(x)$ be the characteristic function of a set E, and $f(x)$ an arbitrary integrable function. Parseval's formula for f and g may be written in the form

---

[1]) We may take for g'' a $(C, 1)$ mean of $\mathfrak{S}[g]$, with index sufficiently large (§ 4.35).

$$\int_E f\,dx = \tfrac{1}{2} a_0 |E| + \sum_{n=1}^{\infty} \int_E (a_n \cos nx + b_n \sin nx)\,dx.$$

Hence $\mathfrak{S}[f]$ *may be integrated term by term over any measurable set and the resulting series is summable* $(C, 1)$ *to the integral of* f *over the set.* As we shall see later, the integrated series converges if $f \in L^r$, $r > 1$. If $f \in L$, this is not necessarily true (§ 4.7.16).

4.42. Applying Parseval's equation 4.4(1) to the functions $f(x+t)$ and $g(x)$, we find the formula

$$(1) \qquad \frac{1}{\pi}\int_0^{2\pi} f(x+t)\,g(x)\,dx = \frac{a_0 a_0'}{2} + \sum_{n=1}^{\infty} \{(a_n a_n' + b_n b_n') \cos nt + (a_n' b_n - a_n b_n') \sin nt\},$$

where the series on the right is uniformly summable $(C, 1)$ in each of the cases considered in Theorem 4.41. Moreover, *given any pair of integrable functions* f, g, *the formula* (1) *holds, in the* $(C, 1)$ *sense, almost everywhere in* t. For the proof it is sufficient to observe that the left-hand side $h(t)$ of (1) is an integrable function and that the series on the right is $\mathfrak{S}[h]$ (§ 2.11).

4.43. Let c_n, c_n' be the complex Fourier coefficients of f, g. The formula 4.4(1) may be written in the form

$$(1) \qquad \frac{1}{2\pi}\int_0^{2\pi} fg\,dx = \sum_{p=-\infty}^{+\infty} c_p c_{-p}' \quad (C, 1).$$

So far we have considered only real functions, but the extension of (1) to the case of f and g complex follows immediately. Substitute $g(x)e^{-inx}$ for $g(x)$ in (1) and let c_p'' denote the Fourier coefficients of $g(x)e^{-inx}$. Since $c_{-p}'' = c_{n-p}'$, we find that

$$(2) \qquad \frac{1}{2\pi}\int_0^{2\pi} fg\,e^{-inx}\,dx = \sum_{p=-\infty}^{+\infty} c_p c_{n-p}' \quad (C, 1), \quad n = 0, \pm 1, \ldots$$

Consequently, *the Fourier series of the product of two functions* f *and* g, $f \in L_\Phi$, $g \in L_\Psi$, *can be obtained by formal multiplication of* $\mathfrak{S}[f]$ *and* $\mathfrak{S}[g]$ *by Laurent's rule. The series defining the coefficients of the product are summable* $(C, 1)$.

The theorem remains valid if $f \in B$, $g \in L$.

4.431. It is obvious that each of the inequalities

$$\sum_{p=-\infty}^{+\infty} |c_p| < \infty, \quad \sum_{p=-\infty}^{+\infty} |c'_p| < \infty$$

implies the absolute convergence of the series in 4.43(2). If both the inequalities are safisfied, then $\mathfrak{S}[fg]$ converges absolutely.

4.432. In Theorems 4.41, 4.42 and 4.43 we may replace summability $(C, 1)$ by (C, k), $k > 0$. The proofs remain the same if we use the results of § 4.37.

4.44. The problem whether summability (C, k) can be replaced by ordinary convergence is more delicate. In Chapter VII we shall prove that the answer is positive if $f \epsilon L^r$, $g \epsilon L^{r'}$, $1 < r < \infty$. This theorem is rather deep; here we will prove a more elementary result. If s_n denotes the n-th partial sum of $\mathfrak{S}[f]$, the difference δ_n between the integral on the left and n-th partial sum of the series on the right in the formula 4.4(1), may be written in the form

$$\text{(1)} \qquad \delta_n = \frac{1}{\pi} \int_0^{2\pi} (f - s_n)\, g\, dx.$$

If the partial sums $s_n(x)$ are uniformly bounded and tend to $f(x)$ almost everywhere, the expression $|f - s_n|\,|g|$ tends to 0 almost everywhere and is majorised by an integrable function. Hence $\delta_n \to 0$, so that the series in 4.4(1) converges to the integral on the left. Hence, reversing the rôle of f and g,

If $f(x)$ is integrable and $g(x)$ is of bounded variation, we have the formula 4.4(1), where the series on the right is convergent[1]).

From this we deduce that, *if f is integrable and periodic,* (α, β) *is a finite interval, and* $g(x)$, $\alpha \leqslant x \leqslant \beta$, *is an arbitrary function of bounded variation, not necessarily periodic, then*

$$\text{(1)} \quad \int_\alpha^\beta fg\, dx = \tfrac{1}{2} a_0 \int_\alpha^\beta g\, dx + \sum_{n=1}^{\infty} \{a_n \int_\alpha^\beta g \cos nx\, dx + b_n \int_\alpha^\beta g \sin nx\, dx\},$$

i. e. Fourier series may be integrated term by term after having been multiplied by any function of bounded variation[2]). In fact, if $\beta - \alpha = 2\pi$, this is nothing else but the previous theorem. The

---

[1]) Young [11].

[2]) The case $g(x) = 1$ has been considered in § 2.621.

case $\beta - \alpha < 2\pi$ may be reduced to the preceding one, putting $g(x) = 0$ for $\beta < x < \alpha + 2\pi$. In the general case we break up the interval (α, β) into a finite number of intervals of length $\leqslant 2\pi$.

4.45 [1]). The last result can be extended to the case of an infinite interval. Without loss of generality we may assume that $(\alpha, \beta) = (-\infty, +\infty)$.

The formula

$$(1)\qquad \int_{-\infty}^{+\infty} fg\,dx = \tfrac{1}{2} a_0 \int_{-\infty}^{+\infty} g(x)\,dx + \sum_{n=1}^{\infty} \{a_n \int_{-\infty}^{+\infty} g \cos nx\,dx + b_n \int_{-\infty}^{+\infty} g \sin nx\,dx\}$$

holds true for any integrable and periodic function f, provided that $g(x)$ is (i) *integrable and* (ii) *of bounded variation over* $(-\infty, +\infty)$. In fact, let us put

$$(2)\qquad G(x) = \sum_{k=-\infty}^{+\infty} g(x + 2k\pi).$$

If the series on the right converges at some point, then it converges uniformly over $(0, 2\pi)$, and its sum $G(x)$ is of bounded variation (§ 2.85). On the other hand, since

$$\sum_{k=-\infty}^{+\infty} \int_0^{2\pi} |g(x + 2k\pi)|\,dx = \int_{-\infty}^{+\infty} |g(x)|\,dx < \infty,$$

we see that the series in (2) has certainly points of convergence (§ 4.2(i)).

Let $c'_n = \tfrac{1}{2}(a'_n - ib'_n)$ be the the Fourier coefficients of $G(x)$. We have then a formula similar to 4.4(1), with g replaced by G. Observing that uniformly convergent series may be integrated term by term after having been multiplied by any integrable function, and remembering that f is periodic, we obtain from (2) that

$$\int_0^{2\pi} fG\,dx = \int_{-\infty}^{+\infty} fg\,dx, \qquad \int_0^{2\pi} G(x)\,e^{-inx}\,dx = \int_{-\infty}^{+\infty} g(x)\,e^{-inx}\,dx,$$

and the formula just referred to takes the form (1). This completes the proof.

The hypothesis that $g(x)$ is integrable over $(-\infty, \infty)$ is, of course, essential for the truth of the equation (1). However, if $a_0 = 0$, condition (i) of the previous theorem may be replaced by the condition that $g(x) \to 0$ as $|x| \to \infty$. In fact, let us put $g^*(x) = g(2k\pi)$ for $2k\pi \leqslant x < 2(k+1)\pi$, $k = 0, \pm 1, \dots$, and let v_k be the total variation of $g(x)$ over $(2k\pi, 2(k+1)\pi)$. The function $g^*(x)$ is of bounded variation and, since $\gamma(x) = g(x) - g^*(x)$ does not exceed v_k in absolute value for $2k\pi \leqslant x < 2(k+1)\pi$, the function $\gamma(x)$ is integrable and of bounded variation over $(-\infty, \infty)$. Let us apply the formula (1) to the functions f and γ. Since the mean value of f over a period is equal to 0, and $g(x) \to 0$ with $1/x$, it is easy to verify that

---

[1]) Hardy [7]. An interesting application to the theory of the Riemann ζ function will be found in Hardy [8].

$$\int_{-\infty}^{+\infty} f\gamma\, dx = \int_{-\infty}^{+\infty} fg\, dx, \quad \int_{-\infty}^{+\infty} \gamma\, e^{-inx}\, dx = \int_{-\infty}^{+\infty} g\, e^{-inx}\, dx,$$

for $n = \pm 1, \pm 2, \ldots$, and the result follows.

4.5. Linear operations. We will now prove a series of results on linear operations[1]. These results will find application in the theory of trigonometrical series.

4.51. Linear and metric spaces. A set E of arbitrary elements will be called a *linear* space if

(i) There exists a commutative and associative operation, denoted by $+$, and called *addition*, applicable to every pair x, y of elements of E. If $x \in E$, $y \in E$, then $x + y \in E$.

(ii) There is an element $o \in E$ (*null* element) such that $x + o = x$ for every $x \in E$.

(iii) There exists a distributive and associative operation, denoted by $\cdot$ and called *multiplication*, applicable to every $x \in E$ and any real number α, with the properties that $1 \cdot x = x$, $0 \cdot x = o$, and that $\alpha \cdot x \in E$.

In most instances it will be convenient to write αx instead of $\alpha \cdot x$. The elements of E will be called points.

E will be called a *metric* space if to every $x \in E$ corresponds a non-negative number $\|x\|$, called the *norm* of x, satisfying the following conditions

$$\|x + y\| \leqslant \|x\| + \|y\|, \quad \|\alpha x\| = |\alpha|\,\|x\|, \quad \|x\| = 0 \text{ is equivalent to } x = o.$$

The *distance* $d(x, y)$ of two points x, y is defined as $\|x - y\|$, where $x - y = x + (-1)\cdot y$. We see that $d(x, y) = d(y, x)$, $d(x, y) \leqslant d(x, z) + d(z, y)$, and that $d(x, y) = 0$ if and only if $x = y$.

We shall say that a sequence of points x_n tends to the limit x, $x \in E$, and write $\lim x_n = x$, or $x_n \to x$, if $\|x - x_n\| \to 0$ as $n \to \infty$.

Once the distance has been defined, we may introduce various notions familiar to the reader from the elements of the theory of point-sets. First of all we define the *sphere* $S(x_0, \rho)$, with centre x_0 and radius ρ, as the set of points x such that $d(x, x_0) \leqslant \rho$.

---

[1]) For a more detailed study we refer the reader to Banach's *Opérations linéaires.*

This notion enables us to introduce various sorts of point-sets: open, closed, non-dense, everywhere dense; furthermore we may consider sets of the first category, i. e. sums of sequences of non-dense sets, and sets of the second category, that is sets which are not of the first category.

4.52. Functional operations. Let us consider besides E another space U which is linear and metric. If to every point $x \in E$ corresponds a point $u = u(x)$ belonging to U, we say that $u(x)$ is a functional operation defined in E. The operation $u(x)$ is said to be *additive* if, for any points x_1, x_2 from E, and any numbers λ_1, λ_2, we have $u(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 u(x_1) + \lambda_2 u(x_2)$. If $u(x_n) \to u(x)$ as $x_n \to x$, we say that u is *continuous* at the point x. If an additive operation $u(x)$ is continuous at some point, it is continuous at any other point, i. e. is continuous everywhere. A necessary and sufficient condition that an additive operation $u(x)$ be continuous is the existence of a number M such that

$$(1) \qquad \|u(x)\| \leqslant M \|x\|, \text{ for every } x \in E.$$

The sufficiency of the condition is obvious. To prove the necessity, let us suppose that there exists a sequence of points x_n such that $\|u(x_n)\| > n\|x_n\|$. Multiplying x_n by a suitable constant we may assume that $\|x_n\| = 1/n$. Then $x_n \to o$, whereas the last inequality gives $\|u(x_n)\| > 1$, so that u would be discontinuous at the point o.

For the sake of brevity, operations that are continuous and additive will be called *linear* operations. The smallest number M satisfying (1) will be denoted by M_u and called the *modulus* of the linear operation u. M_u may be defined as the upper bound of $\|u(x)\|$ on the unit sphere $\|x\| = 1$. It must be remembered that the norms on the right and on the left in (1) may have quite a different meaning, since the spaces E and U may be different. In the applications which we shall consider in this chapter, the space U will be the set R of all real numbers, and $\|u\|$ will be defined as $|u|$.

4.53. Complete spaces. A linear and metric space is said to be *complete*, if for any sequence of points x_n such that $\|x_m - x_n\| \to 0$ as $m, n \to \infty$, there exists a point x such shat $\|x - x_n\| \to 0$. It is an important property of complete spaces that they are of the second category, i. e. cannot be represented as sums of sequences of non-dense sets [1]).

4.54. Examples. In the examples which we consider below the points of E are either real numbers or real functions, and in each case addition and multiplication receive their usual interpretation; the null point will be denoted by 0.

(i) If $E = R$, $\|x\| = |x|$, we have a linear, metric, and complete space.

(ii) If E is the set of all functions $x(t)$ defined and continuous in an interval (a, b), and if $\|x\| = \text{Max}\,|x(t)|$, $a \leqslant t \leqslant b$, then E is a linear, metric, and complete space. The relation $x_n \to x$ means that $x_n(t)$ converges uniformly to $x(t)$.

(iii) If in the previous example we suppose that E is the set of all functions $x(t)$ essentialy bounded on (a, b), and put $\|x\| =$ the essential upper bound of $|x(t)|$, we have again a linear, metric, and complete space; $x_n \to x$ means that $x_n(t)$ converges uniformly to $x(t)$ outside a set T, $|T| = 0$, of values of t.

(iv) Let E be the set of all functions $x(t) \in L^p(a, b)$, $p \geqslant 1$, and let $\|x\| = \|x\|_p = \mathfrak{M}_p[x; a, b]$. The space is linear and metric (§ 4.13). That it is also complete was proved in § 4.2. If $p = \infty$, we obtain, as a special case, the space considered in (iii).

4.541. Classes L^*_Φ. Let Φ and Ψ be a pair of functions complementary in the sense of Young. We ask under what conditions the class $L_\Phi(a, b)$ may be considered as a linear and metric space. First of all we have to define the norm $\|x\|$, and, if the definition is to be useful, the inequality $\|x\| < \infty$ and the integrability of $\Phi[|x(t)|]$ must be, in some degree, equivalent. We might be inclined to put $\|x\| = \Phi_{-1}\left[\int_a^b \Phi(|x|)\,dt\right]$, where Φ_{-1} denotes the function inverse to Φ, but a moment's consideration shows that this definition, which is modelled on the case $\Phi(u) = u^r$, cannot be adopted. First of all the condition $\|x\| = |\alpha|\,\|x\|$ would be satisfied only exceptionally. Moreover, and here lies another difficulty, if $\Phi(u)$ increases very rapidly, the integrability of $\Phi[|x_1(t)|]$ and $\Phi[|x_2(t)|]$ does

[1]) The proofs in the general case and in the case $E = R$ do not differ essentially; see e. g. Hausdorff, *Mengenlehre*, 142.

not involve the integrability of $\Phi[|x_1(t)+x_2(t)|]$. For these reasons we must proceed otherwise[1]).

We shall denote by $L_\Phi^* = L_\Phi^*(a, b)$ the class of all functions $x(t)$, $a \leqslant t \leqslant b$, such that the product $x(t)\,y(t)$ is integrable for every $y(t) \in L_\Psi(a, b)$. If we put

$$\|x\| = \|x\|_\Phi = \operatorname{Sup} \left|\int_a^b x(t)\,y(t)\,dt\right|, \text{ for all } y \text{ with } \rho_y = \int_a^b \Psi(|y|)\,dt \leqslant 1,$$

then it is easy to verify that L_Φ^* is a linear and metric space.

We assume without proof that $\|x\| < \infty$ for every $x \in L_\Phi^*$. This result will be established in § 4.56.

We shall prove that L_Φ^* is also a complete space. Suppose that $\|x_m - x_n\| \to 0$ as $m, n \to \infty$, so that $\|x_m - x_n\| \leqslant \varepsilon$ for m, $n > \nu = \nu(\varepsilon)$.

It follows that

$$(1) \qquad \left|\int_a^b (x_m - x_n)\,y\,dt\right| \leqslant \varepsilon,$$

$$(2) \qquad \int_a^b |x_m - x_n|\,|y|\,dt \leqslant \varepsilon, \text{ if } \rho_y \leqslant 1 \text{ and } m,\ n \geqslant \nu.$$

Let α be the number such that $(b-a)\,\Psi(\alpha) = 1$. Taking $y(t) = \alpha \operatorname{sign}(x_m - x_n)$, we obtain from (1) that $\mathfrak{M}[x_m - x_n; a, b] \leqslant \varepsilon/\alpha$. Since ε is arbitrary, there exists a sequence $\{x_{m_k}(t)\}$ converging almost everywhere to a function $x(t)$ (§ 4.2(ii)), and, applying Fatou's lemma, we obtain from (2) that $\int_a^b |x - x_n|\,|y|\,dt \leqslant \varepsilon$ if $\rho_y \leqslant 1$, and so $\|x - x_n\| \leqslant \varepsilon$ for $n > \nu$. This completes the proof.

We assumed tacitly that $b - a < \infty$, but the theorem holds true if $b - a = \infty$. In fact, proceeding as before, we show that $\mathfrak{M}[x_m - x_n;\ a', b'] \to 0$ for every interval (a', b'), $b' - a' < \infty$, contained in (a, b). Thence we infer the existence of a sequence $\{x_{m_k}(t)\}$ converging almost everywhere in (a, b), and the rest of the proof remains unchanged.

It is obvious that, if $x \in L_\Phi$, then $x \in L_\Phi^*$. The converse is false but we shall prove that, if $x \in L_\Phi^*$, there exists a constant $\theta > 0$ such that $\theta x \in L_\Phi$. More precisely, if $x \in L_\Phi^*$, $x \neq 0$, then

---

[1]) See Orlicz [1].

$\int_a^b \Phi[x/\|x\|]\,dt < 1$. It is sufficient to prove this for x bounded. We will show first that

$$(3) \qquad \left|\int_a^b xy\,dt\right| \leqslant \begin{Bmatrix} \|x\| & \text{if } \rho_y \leqslant 1 \\ \|x\|\rho_y & \text{if } \rho_y > 1 \end{Bmatrix}.$$

The first of these inequalities is obvious; to obtain the second let us replace y by y/ρ_y in the integral on the left. The function Ψ is convex (§ 4.141) and so, by Jensen's inequality, we have $\Psi|y/\rho_y| \leqslant \Psi|y|/\rho_y$, so that

$$\int_a^b \Psi|y/\rho_y|\,dt \leqslant 1, \quad \left|\int_a^b x\frac{y}{\rho_y}\,dt\right| \leqslant \|x\|,$$

and this is just the second inequality (3). From (3) we deduce that the integral on the left does not exceed $\|x\|\rho'_y$ in absolute value, where $\rho'_y = \mathrm{Max}(\rho_y, 1)$.

We know that Young's inequality may degenerate into equality; in particular we have

$$\left|\int_a^b \frac{x}{\|x\|}y\,dt\right| = \int_a^b \Phi\left[\frac{|x|}{\|x\|}\right]dt + \rho_y \leqslant \rho'_y,$$

if $y = \varphi[|x|/\|x\|]\,\mathrm{sign}\,x$ (§ 4.11). Since ρ_y is finite with (a, b), we see that $\rho_y < \rho'_y$, $\rho'_y = 1$ and the result follows[1]).

It is not difficult to see that a necessary and sufficient condition that $x(t)$ should belong to L^*_Φ is the existence of a constant $\theta > 0$ such that $\theta x \in L_\Phi$. In particular, if $\Phi(u)$ satisfies, for large u, the condition $\Phi(2u)/\Phi(u) < C$, where C is independent of u, and if $b - a < \infty$, the classes L_Φ and L^*_Φ are identical. A simple calculation shows that, if $\Phi(u) = u^r$, where $r > 1$, then $\|x\| = r'^{r'}\,\mathfrak{M}_r[x]$, so that, apart from a numerical factor, we have the same norm as in § 4.54 (iv).

4.55. The Banach-Steinhaus theorem. We begin by proving two lemmas.

(i) Let $\{u_n(x)\}$ be a sequence of linear operations which are defined in a linear and metric space E. If F denotes

[1]) Here again the result holds true for $b - a = \infty$.

the set of points for which $\overline{\lim}\,\|u_n(x)\| < \infty$, then $F = F_1 + F_2 + \ldots$, where the sets F_i are closed and the sequence $\{\|u_n(x)\|\}$ is uniformly bounded on each of them.

Let F_{mn} be the set of points where $\|u_m(x)\| \leqslant n$. Since the operations u_m are continuous, the sets F_{mn} are closed, and so are the products $F_n = F_{1n} F_{2n} \ldots$ We have $\|u_m(x)\| \leqslant n$ for $x \in F_n$, $m = 1, 2, \ldots$, and $F = F_1 + F_2 + \ldots$

(ii) If the space E of the previous lemma is complete, and the set F of the second category (in particular, if $F = E$), then there exists a sphere $S(x_0, \rho)$, $\rho > 0$, and a number K, such that $\|u_m(x)\| \leqslant K$ for $x \in S(x_0, \rho)$ and $m = 1, 2, \ldots$

Since $F = F_1 + F_2 + \ldots$, and F is of the second category, at least one of the sets $F_1, F_2, \ldots$, say F_K, is not non-dense and so there exists a sphere $S(x_0, \rho)$ in which F_K it everywhere dense. Since F_K is closed, we have $S(x_0, \rho) \subset F_K$, and consequently $\|u_m(x)\| \leqslant K$ for $x \in S(x_0, \rho)$, $m = 1, 2, \ldots$

Let $\{u_n(x)\}$ be a sequence of linear operations defined in a linear, metric, and complete space E, and let M_{u_n} denote the modulus of the operation u_n (4.52). If $\overline{\lim}\,\|u_n(x)\|$ is finite for every point x belonging to a set F of the second category in E, then the sequence M_{u_n} is bounded. In other words, there is a constant M such that $\|u_m(x)\| \leqslant M\,\|x\|$, $m = 1, 2, \ldots$ [1].

Let $S(x_0, \rho)$ be the sphere considered in (ii). Since every $x \in S(0, \rho)$ can be writen in the form $x = x_1 - x_0$, where $x_1 \in S(x_0, \rho)$, we see that $\|u_n(x)\| \leqslant 2K$ for $x \in S(0, \rho)$, $n = 1, 2, \ldots$ It follows that $\|u_n(x)\|/\|x\| \leqslant 2K/\rho = M$ on the sphere $\|x\| = \rho$, and so $\|u_n(x)\| \leqslant M\,\|x\|$ for every x and n.

The theorem may also be stated as follows. If the sequence $\|u_n(x)\|$ is unbounded at some point, the set of points where this sequence is bounded is of the first category in E.

4.56. Corollaries. In this section we consider operations of the form

$$u(x) = \int_a^b x(t)\,y(t)\,dt, \tag{1}$$

[1]) Banach and Steinhaus [1]. The idea of the proof, due to Saks, may be applied to many similar problems.

where x belongs to a linear, metric, and complete space E, and y is a function such that xy is integrable for every $x \in E$.

(i) If the integral (1) is defined for every bounded, or even only continuous, function $x(t)$, then $y \in L(a, b)$. (ii) Conversely, if the integral (1) converges for every $x \in L(a, b)$, then the function y is essentially bounded. (iii) If the integral (1) exists for every $x \in L^*_\Phi(a, b)$, then $y \in L^*_\Psi(a, b)$, where Φ and Ψ are functions complementary in the sense of Young.

To avoid repetition we take these theorems for granted; they can be deduced from more general results which we will now prove.

(iv) *If the sequence*

$$(2) \qquad u_n(x) = \int_a^b x(t)\, y_n(t)\, dt$$

is bounded for every bounded, or even only continuous, function x, then $\mathfrak{M}[y_n; a, b] = O(1)$. (v) *If $\{u_n(x)\}$ is bounded for every $x \in L(a, b)$, then the essential upper bounds of y_n are uniformly bounded.* (vi) *If $\{u_n(x)\}$ is bounded for every $x \in L^*_\Phi$, then $\|y_n\|_\Psi = O(1)$.*

To prove (iv), we observe that, in virtue of (i), each of the functions y_n is integrable, and so $u_n(x)$ is a linear operation defined in the space considered in § 4.54(iv), $r = 1$. Putting $x = \operatorname{sign} y_n$, we see that the modulus M_{u_n} of the operation u_n is equal to $\mathfrak{M}[y_n]$, and it is sufficient to apply the Banach-Steinhaus theorem. The case of continuous functions is not essentially different: we consider the space of § 4.54(iii), and, since the function $\operatorname{sign} y_n(t)$ is the limit of a bounded and almost everywhere convergent sequence of continuous functions, we have $M_{u_n} = \mathfrak{M}[y_n]$ again.

In case (v) we proceed similarly: each of the functions y_n is essentially bounded, and $M_{u_n} =$ the essential upper bound of $|y_n|$.

In case (vi) each of the functions y_n belongs (by (iii)) to L^*_Ψ. In virtue of the inequality $\lambda\, |u_n(x) - u_n(x_0)| \leqslant \|x - x_0\|_\Phi\, \rho'_{\lambda y_n}$ (§ 4.541), where $\lambda > 0$ is a constant so small that $\lambda y_n \in L_\Psi$, we obtain that $u_n(x)$ is a linear operation. Hence, by Theorem 4.55, $|u_n(x)| \leqslant M \|x\|_\Phi$, for $n = 1, 2, \ldots$ Now, if the integral of $\Phi(|x|)$ over (a, b) does not exceed 1, then $\|x\| \leqslant 2$, and so the inequality $|u_n(x)| \leqslant M \|x\|_\Phi$ gives $\|y_n\|_\Psi \leqslant 2M$, $n = 1, 2, \ldots$, and the theorem is established.

The above proof may be used to establish (i), (ii) and (iii) (proposition (i) in the case of bounded functions, is trivial, since we may put $x = \operatorname{sign} y$). To prove (iii) we put $y_n(t) = y(t)$ whenever $|y| \leqslant n$, and $y_n(t) = 0$ elsewhere. The formula (2) defines a sequence of linear operations, and the inequality $\|y_n\|_\Psi = O(1)$ implies $\|y\|_\Psi < \infty$.

(vii) *If the sequence* (2) *is bounded for every* $x \in L^r$, *then* $\mathfrak{M}_{r'}[y_n] = O(1)$ [1]. (viii) *If the sequence* (2) *is bounded for every* $x \in L_\Phi$, *then there exists a constant* $\theta > 0$ *such that* $\mathfrak{M}[\Psi|\theta y_n|] = O(1)$ [2].

The first of these propositions is a corollary of (vi). To obtain the second we observe that, if $\|y_n\|_\Psi \leqslant M$ for $n = 1, 2, \ldots$, then $\mathfrak{M}[\Psi|y_n/M|] < 1$. (§ 4.541).

The theorems which we have established for integrals have analogues for infinite sums. The proofs remain unchanged [3].

4.6. Transformations of Fourier series. Given a numerical sequence $\lambda_0, \lambda_1, \lambda_2, \ldots$, let us consider, besides the series

$$\tag{1} \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

the following two series

$$\tag{2} \tfrac{1}{2} \lambda_0 + \sum_{n=1}^{\infty} \lambda_n \cos nx,$$

$$\tag{3} \tfrac{1}{2} a_0 \lambda_0 + \sum_{n=1}^{\infty} \lambda_n (a_n \cos nx + b_n \sin nx).$$

Given two classes P, Q of trigonometrical series we shall denote by (P, Q) the class of sequences $\{\lambda_n\}$ transforming P into Q, that is such that, whenever (1) belongs to P, (3) belongs to Q [4].

---

[1] Hahn [1].

[2] Birnbaum and Orlicz [1].

[3] See e. g. Banach, *Opérations linéaires.*

[4] For the problems discussed in this paragraph see Young [9], Steinhaus [2], [3], Szidon [1], Fekete [1], M. Riesz [3], Zygmund [3], Bochner [1], Verblunsky [1], Kaczmarz [5], Hille and Tamarkin [1₂].

A necessary and sufficient condition for $\{\lambda_n\}$ to belong to any one of the classes (B, B), (C, C), (L, L), (S, S) is that the series (2) *should be a Fourier-Stieltjes series.*

Let (1) be a $\mathfrak{S}[f]$ and let $\sigma_n(x)$, $l_n(x)$, $\sigma_n^*(x)$ denote the $(C, 1)$ means of the series (1), (2), (3) respectively. We have

$$\sigma_n^*(x) = \frac{1}{\pi}\int_0^{2\pi} f(x+t)\, l_n(t)\, dt. \tag{4}$$

Put $x = 0$. If $\lambda_n \in (C, C)$, or if $\lambda_n \in (B, B)$, the sequence $\{\sigma_n^*(0)\}$ is bounded for every $f \in C$, and, by Theorem 4.56 (iv), we have $\mathfrak{M}[l_n] = O(1)$, i. e. (2) belongs to S. Conversely, if the series (2) is a $\mathfrak{S}[dL] \in S$, the formula (4) may be written in the form

$$\sigma_n^*(x) = \frac{1}{\pi}\int_0^{2\pi} \sigma_n(x+t)\, dL(t). \tag{5}$$

Thence we deduce that the uniform boundedness of $\{\sigma_n(x)\}$ involves that of $\{\sigma_n^*(x)\}$. Similarly, if $\sigma_m(x) - \sigma_n(x)$ tends uniformly to 0 as $m, n \to \infty$, so does $\sigma_m^*(x) - \sigma_n^*(x)$, and this completes the proof of the theorem as regards the classes (B, B) and (C, C).

If $\{\lambda_n\} \in (S, S)$, it transforms, in particular, the series $\frac{1}{2} + \cos x + \cos 2x + \ldots \in S$ into the series (2), which must, therefore, belong to S. Conversely, if the series (2) is a $\mathfrak{S}[dL]$, we obtain from (5) that

$$|\sigma_n^*(x)| \leqslant \frac{1}{\pi}\int_0^{2\pi} |\sigma_n(x+t)|\,|dL(t)|. \tag{6}$$

Integrating this inequality over $(0, 2\pi)$, and inverting the order of integration on the right, we obtain that $\mathfrak{M}[\sigma_n^*] \leqslant (v/\pi)\,\mathfrak{M}[\sigma_n]$, where v is the total variation of $L(t)$ over $(0, 2\pi)$. Hence the series (3) belongs to S.

It remains only to consider the case (L, L). Since

$$|\sigma_m^*(x) - \sigma_n^*(x)| \leqslant \frac{1}{\pi}\int_0^{2\pi} |\sigma_m(x+t) - \sigma_n(x+t)|\,|dL(t)|,$$

$$\mathfrak{M}[\sigma_m^* - \sigma_n^*] \leqslant (v/\pi)\,\mathfrak{M}[\sigma_m - \sigma_n],$$

the sufficiency of the condition is obvious (§ 4.34). To prove the necessity let us consider, for every n, a system $I_n = \{(\alpha_1^n, \beta_1^n), (\alpha_2^n, \beta_2^n), \ldots\}$ of non-overlapping intervals. It follows from (4) that

(7) $$\int_{I_n} \sigma_n^*(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} f(t)\left\{\int_{I_n} l_n(t-x)\,dx\right\}dt.$$

Suppose that (2) does not belong to S, so that the indefinite integrals of the functions $l_n(x)$ are not of uniformly bounded variation. We can then find a sequence $I_1, I_2, \dots$ such that the coefficient of $f(t)$ in (7) is not uniformly bounded. By Theorem 4.56(v), there is an integrable f such that the right-hand side in (7) is unbounded, and, à fortiori, $\mathfrak{M}[\sigma_n] \neq O(1)$. It follows that the series (3) does not belong to S, and, in particular, does not belong to L, although (1) is a Fourier series.

4.61. Let $\bar{P}$ denote the class of trigonometrical series conjugate to those belonging to P. It is plain that if P, and similarly Q, is one of the classes B, C, L, S, then $(\bar{P}, \bar{Q}) = (P, Q)$.

A necessary and sufficient condition that $\{\lambda_n\}$ *should belong to any one of the classes* $(\bar{B}, B)$, $(\bar{C}, C)$, $(\bar{L}, L)$, $(\bar{S}, S)$ *is that the series conjugate to* 4.6(2) *should belong to* S.

The proof is similar to that of Theorem 4.6. We need only slightly change the formulae which we have used, so as to introduce conjugate series. In fact, let $\bar{\sigma}_n(x)$ and $\bar{\sigma}_n^*(x)$ denote the first arithmetic means of the series

(1) $\sum_{n=1}^{\infty}(a_n \sin nx - b_n \cos nx)$, (2) $\sum_{n=1}^{\infty}\lambda_n(a_n \sin nx - b_n \cos nx)$

respectively, and let $\bar{l}_n(x)$ be the arithmetic means of the series $\lambda_1 \sin x + \lambda_2 \sin 2x + \dots$, conjugate to 4.6(2). If the series 4.6(1) is a $\mathfrak{S}[f]$, we have the formula

(3) $$\bar{\sigma}_n^*(x) = -\frac{1}{\pi}\int_0^{2\pi} f(x+t)\,\bar{l}_n(t)\,dt,$$

analogous to 4.6(4). Considering, for example, the case (B, B), we suppose that the series 4.6(1) belongs to B and ask under what conditions (2) is the Fourier series of a bounded function. Arguing as in the preceding section, we obtain that the necessary and sufficient condition is $\mathfrak{M}[\bar{l}_n] = O(1)$. The remaining cases may be left to the reader.

4.62. Let $\chi(u)$, $u \geqslant 0$, be a function non-negative, convex, bounded in any finite interval, and tending to infinity with u.

If the series 4.6(1) *is the Fourier series of a function f such that $\chi(|f|)$ is integrable, and if* 4.6(2) *is a* $\mathfrak{S}[dL]$, *then* 4.6(3) *is the Fourier series of a function $g(x)$ such that $\chi(|g|\pi/v)$ is integrable, where v denotes the total variation of L over* $(0, 2\pi)$.

Without real loss of generality we may suppose that $\chi(u)$ is non-decreasing. Let $t_i = 2\pi i/N$, $i = 0, 1, \ldots, N$, and let $v(x)$ denote the total variation of L over $(0, x)$, so that $v(2\pi) = v$. Dividing both sides of the inequality 4.6(6) by v, and applying the mean-value theorem in each of the intervals (t_{i-1}, t_i), we obtain that

$$\pi\,|\sigma_n^*(x)|/v \leqslant \sum_{i=1}^{N} \xi_i\, p_i \Big/ \sum_{i=1}^{N} p_i,$$

where $p_i = v(t_i) - v(t_{i-1})$, $\xi_i = \sigma_n(x + t_i')$, $t_{i-1} \leqslant t_i' \leqslant t_i$. Applying Jensen's inequality, and making $N \to \infty$, we obtain that

$$\chi\left\{\frac{\pi}{v}\,|\sigma_n^*(x)|\right\} \leqslant \sum_{i=1}^{N} \chi(\xi_i)\, p_i \Big/ \sum_{i=1}^{N} p_i, \quad \chi\left\{\frac{\pi}{v}\,|\sigma_n^*(x)|\right\} \leqslant \frac{1}{v}\int_0^{2\pi} \chi|\sigma_n(x+t)|\,|dL(t)|.$$

Now it is sufficient to integrate the last inequality over $(0, 2\pi)$, to invert the order of integration on the right, and to apply Theorem 4.33.

It must be emphasized that the condition which we imposed upon the series 4.6(2) is only sufficient and by no means necessary. This is easily seen in the case $\chi(u) = u^2$, since, by the Riesz-Fischer theorem, a sequence $\{\lambda_n\}$ belongs to the class (L^2, L^2) if and only if $\lambda_n = O(1)$.

The theorem which we have proved may also be stated in the following form. If $\Phi(x)$ is a Young function and the series 4.6(2) belongs to S, the sequence $\{\lambda_n\}$ belongs to the class (L_Φ^*, L_Φ^*). It belongs in particular to every class (L^r, L^r), $r > 1$.

4.63. Let Φ, Ψ and Φ_1, Ψ_1 be two pairs of Young's complementary functions.

The classes $(L_\Phi^*, L_{\Phi_1}^*)$ and $(L_{\Psi_1}^*, L_\Psi^*)$ are identical.

The proof will be based on the following lemma. *A necessary and sufficient condition that the series* 4.6(1) *should be a* $\mathfrak{S}[f]$ *with* $f \in L_\Phi^*$ *is that, for every* $g \in L_\Psi^*$ *with Fourier coefficients* a_n', b_n', *the series*

(1) $$\frac{a_0 a'_0}{2}+\sum_{n=1}^{\infty}(a_n a'_n+b_n b'_n)$$

should be finite $(C, 1)$[1].

If $f \in L^*_\Phi$, $g \in L^*_\Psi$, there exist two constants $\lambda > 0$ and $\mu > 0$ such that $\lambda f \in L_\Phi$, $\mu g \in L_\Psi$, and the necessity of the condition follows from Theorem 4.41(i). To prove that the condition is sufficient let $\sigma_n(x)$ and τ_n denote the first arithmetic means of the series 4.6(1) and (1) respectively. We have then

$$\tau_n = \frac{1}{\pi}\int_0^{2\pi} g(t)\,\sigma_n(t)\,dt.$$

Since the sequence $\{\tau_n\}$ is bounded for every $g \in L^*_\Psi$, it follows that $\|\sigma_n\|_\Phi = O(1)$, which shows that the series 4.6(1) belongs to L^*_Φ (§§ 4.56(vi), 4.33).

Now it is easy to prove the theorem. If $\{\lambda_n\} \in (L^*_\Phi, L^*_{\Phi_1})$ then, for every $f \in L^*_\Phi$ with Fourier coefficients a_n, b_n, and every $g \in L^*_{\Psi_1}$ with Fourier coefficients a'_n, b'_n, the series

(2) $$\tfrac{1}{2}\lambda_0 a_0 a'_0 + \sum_{n=1}^{\infty}(\lambda_n a_n a'_n + \lambda_n b_n b'_n)$$

is finite $(C, 1)$.

It means, in virtue of the lemma, that the series with coefficients $\lambda_n a'_n$, $\lambda_n b'_n$ belongs to L^*_Ψ, i. e. $\{\lambda_n\} \in (L^*_{\Psi_1}, L^*_\Psi)$.

Corollaries. (i) If Φ and Ψ are complementary functions, the classes (L^*_Φ, L^*_Φ) and (L^*_Ψ, L^*_Ψ) are identical.

(ii) If $r > 1$, $s > 1$, the classes (L^r, L^s) and $(L^{s'}, L^{r'})$ are identical. In particular $(L^r, L^r) = (L^{r'}, L^{r'})$.

In Ch. IX we shall prove that, if $r < s < r'$, the class (L^r, L^r) is contained in (L^s, L^s).

4.64. *If the series* 4.6(2) *belongs to* L, *then* $\{\lambda_n\} \in (S, L)$, $\{\lambda_n\} \in (B, C)$. Let 4.6(1) be a $\mathfrak{S}[dF]$. From the formula 4.6(4), with $f(x+t)$ replaced by $dF(x+t)$, we find that $\pi\mathfrak{M}[\sigma^*_m - \sigma^*_n]$ does not exceed $\mathfrak{M}[l_m - l_n]$ multiplied by the total variation of F over $(0, 2\pi)$. It follows that $\mathfrak{M}[\sigma^*_m - \sigma^*_n] \to 0$ as $m, n \to \infty$. Thus

---

[1]) A series $u_0 + u_1 + \dots$ is said to be finite (C, r), if the r-th Cesàro means of the series forms a bounded sequence.

the series 4.6(3) belongs to L. Similarly we find from 4.6(4) that $\pi|\sigma_m^* - \sigma_n^*|$ does not exceed $\mathfrak{M}[l_m - l_n]$. Max $|f|$, i. e. B is transformed into C.

A similar proof shows that, *if the series conjugate to* 4.6(2) *belongs to* L, *then* $\{\lambda_n\} \in (\bar{B}, C)$, $\{\lambda_n\} \in (\bar{S}, L)$.

4.65. The conditions which we imposed upon $\{\lambda_n\}$ in the preceding section are not only sufficient but also necessary. For the first parts of the theorems this follows immediately by considering the series $\frac{1}{2} + \cos x + \cos 2x + \ldots \subset S$ and $\sin x + \sin 2x + \ldots \in \bar{S}$. For the second parts the proof is more difficult and we do not propose to consider it here.

Let $\{\lambda_n\}$ be an arbitrary convex sequence tending to 0, e. g. $\lambda_n = n^{-\alpha}$, $\alpha > 0$, $\lambda_n = 1/\log n$, $\lambda_n = 1/\log\log n$, for n sufficiently large. In § 5.12 we shall prove that the series 4.6(2) with such coefficients belongs to L, i. e. $\{\lambda_n\}$ transforms Fourier-Stieltjes series into Fourier series, bounded functions into continuous.

The sequence $\lambda_n = 1/(\log n)^{1+\varepsilon}$, $\varepsilon > 0$, $n > 1$, belongs to $(\bar{S}, L)$ and $(\bar{B}, C)$. For $\varepsilon = 0$ this is no longer true (§ 5.13).

4.7. Miscellaneous theorems and examples.

1. Let $\varphi(x)$, $x \geqslant 0$, be convex, increasing to ∞ with x, and vanishing at the origin. If $\psi(y)$ is the inverse function, and $a \geqslant 0$, $b \geqslant 0$, then $ab \leqslant a\varphi(a) + b\psi(b)$.

2. Given a function $F \in L^r(a,b)$, $r > 1$, let $I_G = |\int_a^b FG\,dx|$, where $G \in L^{r'}$. Show that $\mathfrak{M}_r[F] = \text{Sup}\, I_G$ for all G with $\mathfrak{M}_{r'}[G] \leqslant 1$.

[Since $\mathfrak{M}_r[F] < \infty$, we may suppose that $\mathfrak{M}_r[F] = 1$. By Young's inequality we have $I_G \leqslant \mathfrak{M}_r^r[F]/r + \mathfrak{M}_{r'}^{r'}[G]/r' \leqslant 1$, and for a special function G, viz. when $G = |F|^{r-1}$ sign F, we have $I_G = 1$. It is easy to see that the theorem holds true when $\mathfrak{M}_r[F] = \infty$.

We add that, if $(a, b) = (0, 2\pi)$, it is sufficient to take for the functions G only trigonometrical polynomials, since for any $G \in L^{r'}$ we can find a trigonometrical polynomial g such that $\mathfrak{M}_{r'}[G - g] < \varepsilon$ and so, by Minkowski's inequality, $|\mathfrak{M}_{r'}[G] - \mathfrak{M}_{r'}[g]| < \varepsilon$].

3. Let $\chi(x)$, $x \geqslant 0$, be convex and strictly increasing, $\chi(0) = 0$. Let $f(x)$ be integrable and periodic, and $F(x)$ the indefinite integral of $f(x)$. If $\mathfrak{M}[\chi|f|; 0, 2\pi] \leqslant C$, and $0 < h \leqslant 2\pi$, then $|F(x+h) - F(x)| \leqslant h\chi_{-1}(C/h)$, where χ_{-1} is the function inverse to χ. If $f \in L^r$, $r \geqslant 1$, then $\omega(F; \delta) = o(\delta^{1/r'})$. Young [3].

[Apply Jensen's inequality].

4. If $f\log^{+}|f|$ is integrable over $(-\pi,\pi)$ [1]), so is $f\log 1/|x|$.
[Apply Young's inequality to the product $2|f|\cdot\frac{1}{2}\log 1/|x|$].

5. If a_n are the cosine coefficients of $f(x)$, and $f(x)\log 1/|x|$ is integrable over $(-\pi,\pi)$, the series $a_1+a_2/2+a_3/3+\dots$ converges and has the sum $-\frac{1}{\pi}\int_0^{2\pi} f(x)\log(2\sin\frac{1}{2}x)\,dx$. Hardy and Littlewood [7].

[Express the partial sums of the series as an integral. The partial sums of the series $\cos x+\frac{1}{2}\cos 2x+\dots$ are $O(\log 1/|x|)$ uniformly in n. This follows from the first formula 1.12(3) and from the general theorem that, if $u_n=O(1/n)$, $f(r)=u_0+u_1r+u_2r^2+\dots$, $s_n=u_0+\dots+u_n$, then $f(r)-s_n=O(1)$ as $r=1-1/n\to 1$. To prove the latter fact we observe that, if $|u_k|\leqslant A/k$, then $|f(r)-s_n|\leqslant(1-r)[|a_1|+2|a_2|+\dots+n|a_n|]+A/n(1-r)=O(1)$].

6. Let $\omega_p(\delta)=\omega_p(\delta;f)=\operatorname{Max}\mathfrak{M}_p[f(x+h)-f(x);\,0,2\pi]$ for $0<h\leqslant\delta$. The function f is said to belong to $\operatorname{Lip}(\alpha,p)$, if $\omega_p(\delta)=O(\delta^{\alpha})$. Show that (i) if $f\in\operatorname{Lip}(\alpha,p)$, then $f\in\operatorname{Lip}(\alpha,p_1)$, $0<p_1<p$, (ii) if f is continuous and $p\to\infty$, then $\omega_p(\delta)\to\omega(\delta)$, (iii) if $f\in\operatorname{Lip}(\alpha,p)$, then $f\in L^p$.

[To prove (iii), integrate the inequality $\mathfrak{M}_p^p[f(x+h)-f(x)]\leqslant C$ with respect to h, invert the order of integration, and consider a value of x for which the function $[f(x+h)-f(x)]^p$ is integrable with respect to h. Tamarkin, *Fourier Series*, p. 49].

7. A necessary and sufficient condition that $f(x)$ should belong to $\operatorname{Lip}(1,1)$ is that there should exist a function $g(x)$ of bounded variation, equivalent to $f(x)$. Hardy and Littlewood [6_1].

[To prove that the condition is sufficient, let $\sigma_n(x)$ be the first arithmetic means of $\mathfrak{S}[f]$. Then $\mathfrak{M}[\sigma_n(x+h)-\sigma_n(x)]\leqslant\mathfrak{M}[f(x+h)-f(x)]\leqslant Ch$, $\mathfrak{M}[\sigma_n'(x)]\leqslant C$, and it is sufficient to apply Theorem 4.325. To prove that the condition is necessary it is enough to suppose that $f(x)$ is non-decreasing.
For a more elementary proof see the paper referred to above].

8. A necessary and sufficient condition that $f(x)$ should belong to $\operatorname{Lip}(1,p)$, $p>1$, is that f should be equivalent to the indefinite integral of a function belonging to L^p. Hardy and Littlewood [6_1].
[The condition is necessary since

$$\mathfrak{M}_p^p[f(x+h)-f(x)]\leqslant\int_0^{2\pi}\Big\{\int_x^{x+h}|f'(t)|\,dt\Big\}^p dx\leqslant h^p\int_0^{2\pi}|f'|^p\,dt.$$

To show that the condition is sufficient we prove that $\mathfrak{M}_p[\sigma_n'(x)]=O(1)$].

9. Let $s_n(x)$ and $\sigma_n(x)$ be the partial sums and the $(C,1)$ means of $\mathfrak{S}[f]$. Then (i) a necessary and sufficient condition that f should belong to $\operatorname{Lip}\alpha$, $0<\alpha\leqslant 1$, is that the σ_n should belong to $\operatorname{Lip}\alpha$ uniformly in n, (ii) if $f\in\operatorname{Lip}\alpha$, then $\omega(\delta;s_n)=O(\delta^{\alpha}\log 1/\delta)$ uniformly in n.

---

[1]) If u is real, u^{+} denotes the number $\operatorname{Max}(u,0)$.

10. Let $\sigma_n(x)$ be the first arithmetic means of a trigonometrical series. A necessary and sufficient condition that the series should be a Fourier series is that there should exist a function $\varphi(u) \geqslant 0$, $\varphi(u)/u \to \infty$ with u, and such that $\mathfrak{M}[\varphi|\sigma_n|] = O(1)$. de la Vallée-Poussin [2].

[If $\varphi(u) \geqslant 0$, $\varphi(u)/u \to \infty$, there exists a convex function $\varphi_1(u) \leqslant \varphi(u)$, satisfying the same conditions. If $f \in L$, then there exists a function $\varphi(u)$, $\varphi(u)/u \to \infty$, such that $\varphi(|f|) \in L$].

11. Let $f(r, x) = \frac{1}{2}a_0 + (a_1 \cos x + b_1 \sin x) r + \dots$ A necessary and sufficient condition that $f(r, x)$ should be a difference of two non-negative harmonic functions is that $\mathfrak{M}[|f(r, x)|] = O(1)$ as $r \to 1$.

12. Let $\varphi(u)$ be convex, non-negative, and increasing, and let $\frac{1}{2}a_0 + (a_1 \cos x + b_1 \sin x) + \dots$ be a $\mathfrak{S}[dF]$. A necessary and sufficient condition that the positive variation $P(x)$ of $F(x)$ should be absolutely continuous, and that $P'(x)$ should belong to L_φ is: $\mathfrak{M}[\varphi\{f^+(r, x)\}] = O(1)$ as $r \to 1$, where $f(r, x)$ has the same meaning as in the previous theorem.

13. If $f \in L^2$, and c_n are the complex Fourier coefficients of f, then the function $h(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x+t) f(t)\, dt$ is continuous, and $h(x) \sim \sum_{n=-\infty}^{+\infty} |c_n|^2 e^{inx}$ (§ 2.12). Show that Parseval's theorem is a simple corollary of this result.

14. Let $\sigma_n^r(x)$, $r > 0$, be the r-th Cesàro means of a trigonometrical series. A necessary and sufficient condition that the series should belong to L_Φ^* is $\|\sigma_n^r\|_\Phi = O(1)$. If the series is a $\mathfrak{S}[f] \in L_\Phi^*$ then $\|f - \sigma_n^r\|_\Phi \to 0$ as $n \to \infty$.

15. Let X be the set of all functions $x(t)$ which are the characteristic functions of measurable sets contained in $(0, 2\pi)$. If the sequence 4.56(2), where $(a, b) = (0, 2\pi)$, is bounded for every $x \in X$, then $\mathfrak{M}[y_n] = O(1)$. Saks [1].

[The proof runs on the same lines as that of § 4.55. If we put $\|x_1 - x_2\| = \int_0^{2\pi} |x_1(t) - x_2(t)|\, dt$, X becomes a metric and complete space. X is not a linear space but it has the following property which may in most cases be used instead of linearity: let $S(u, \rho)$, $\rho > 0$, be an arbitrary sphere; for any $x \in S(0, \rho)$ there exist two points x_1 and x_2 belonging to $S(u, \rho)$ such that $x = x_1 - x_2$. It suffices to put $x_1(t) = u(t) + x(t)\,[1 - u(t)]$, $x_2(t) = u(t)\,[1 - x(t)]$].

16. There exists a function $f \in L$ and a measurable set E such that $\mathfrak{S}[f]$ integrated formally over E diverges.

[This follows from the previous theorem and from the results of § 5.12].

CHAPTER V.

Properties of some special series.

5.1. In this chapter we intend to study some particular series, which are not only interesting in themselves, but provide examples illuminating many points of the general theory. The latter consideration will be decisive in our choice of material.

5.11. Series with coefficients monotonically tending to zero. In § 1.23 we have proved that if a sequence $\{a_n\}$ decreases monotonically to 0, or, more generally, if $\{a_n\}$ tends to 0 and is of bounded variation, both series

$$(1) \qquad a) \;\; \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx, \qquad b) \;\; \sum_{n=1}^{\infty} a_n \sin nx$$

converge uniformly, except in arbitrarily small neighbourhoods of the points $x = 0 \pmod{2\pi}$. We will now prove some further theorems on the behaviour of these series.

It is obvious that, if $a_n \geqslant 0$, a necessary and sufficient condition for the uniform convergence of the series (1a) is the convergence of $a_0 + a_1 + \dots$ For the series (1b) the situation is less trivial.

If $a_n \geqslant a_{n+1} \to 0$, *a necessary and sufficient condition for the uniform convergence of the series* (1b) *is* $na_n \to 0$ [1]).

We shall consider only the values $0 < x \leqslant {}^1/_4\, \pi$. To prove the sufficiency we denote by $r_M(x)$ the M-th remainder $a_M \cos Mx + \dots$ of the series (1b), and put $\varepsilon_n = \operatorname{Max} ka_k$ for $k \geqslant n$, $N = N_x = [1/x] + 1$, so that $N > 1$, $1/N < x \leqslant 1/(N-1)$. For any x we put $r_M(x) =$

[1]) Chaundy and Jollife [1].

$= r'_M(x) + r''_M(x)$, where r''_M denotes the sum of all the terms belonging to r_M with indices $\geqslant N$. If $N \leqslant M$, we have $r'_M(x) = 0$. If $N > M$, then

$$|r'_M(x)| \leqslant x \sum_{k=M}^{N-1} k a_k \leqslant \frac{1}{(N-1)} \varepsilon_M (N-M) \leqslant \varepsilon_M.$$

It follows that $|r'_M(x)| \leqslant \varepsilon_M$ for every $M > 0$. Applying Abel's transformation to r''_M, $M < N$, we obtain

$$|r''_M(x)| \leqslant \sum_{k=N}^{\infty} (a_k - a_{k+1}) |\bar{D}_k(x)| + a_N |\bar{D}_{N-1}(x)| \leqslant \frac{8a_N}{x} \leqslant 8Na_N \leqslant 8\varepsilon_M,$$

since $|\bar{D}_k(x)| = |\sin x + \dots + \sin kx| \leqslant 1/\sin\frac{1}{2}x \leqslant \pi/x \leqslant 4/x$. Similarly, if $N \leqslant M$, then $|r''_M(x)| \leqslant 8\, a_M/x \leqslant 8\, Ma_M \leqslant 8\,\varepsilon_M$. Hence $|r_M(x)| \leqslant |r'_M(x)| + |r''_M(x)| \leqslant 9\varepsilon_M$ for $0 < x \leqslant {}^1/_4\,\pi$. Since this inequality is obvious for $x = 0$, the uniformity of convergence follows.

Conversely, assuming that the series (1b) converges uniformly, and putting $x = \pi/2N$, $N \to \infty$, we deduce from the inequality

$$\sum_{[{}^1/_2 N]+1}^{N} a_n \sin nx \geqslant \sin\frac{\pi}{4} \cdot a_N \sum_{[{}^1/_2 N]+1}^{N} 1 \geqslant \sin\frac{\pi}{4} \cdot \tfrac{1}{2} N a_N$$

that $Na_N \to 0$. This completes the proof.

If na_n is bounded, the above argument shows that the partial sums $s_n(x)$ of (1b) are uniformly bounded, but, as is seen from the series $\sin x + \frac{1}{2}\sin 2x + \dots$, the sequence $\{s_n(x)\}$ need not be uniformly convergent.

5.12 [1]. (i) *If* $a_n \to 0$ *and* $\{a_n\}$ *is quasi-convex, the series* 5.11(1a) *converges, save for* $x = 0$, *to an integrable function* $f(x)$, *and is the Fourier series of* $f(x)$. *If* $\{a_n\}$ *is convex,* $f(x)$ *is non-negative.*

Applying Abel's transformation twice, we obtain the expression for the n-th partial sum of the series 5.11(1a)

$$(1)\quad s_n(x) = \sum_{m=0}^{n} (m+1)\, \Delta^2 a_m K_m(x) + K_n(x)(n+1)\, \Delta a_{n+1} + D_n(x) a_{n+1},$$

where D_m and K_m denote Dirichlet's and Fejér's kernels. If $x \neq 0$, the last two terms on the right tend to 0 with $1/n$, and therefore $s_n(x) \to f(x) = \Delta^2 a_0 K_0(x) + 2\Delta^2 a_1 K_1(x) + \dots$, which is non-negative for $\{a_n\}$ convex. Since $|f(x)| \leqslant |\Delta^2 a_0| K_0(x) + 2|\Delta^2 a_1| K_1(x) + \dots$,

---

[1] Young [9], Kolmogoroff [1].

and the last series integrated over $(-\pi, \pi)$ gives the finite value $\pi(|\Delta^2 a_0|+2|\Delta^2 a_1|+...)$, $f(x)$ is integrable.

The problem of the series 5.11(1a) being a Fourier series is slightly more delicate, and we shall see in a moment why it is so.

From the expression for $f(x)$ and $s_n(x)$ we easily find that $|f(x)-s_n(x)|$ is contained between the expressions

$$\pm\{\sum_{m=n+1}^{\infty}(m+1)|\Delta^2 a_m| K_m(x)+K_n(x)(n+1)|\Delta a_{n+1}|\}+a_{n+1}|D_n(x)|.$$

Integrating this over $(-\pi, \pi)$ we find that $\mathfrak{M}[f-s_n]=o(1)+2a_{n+1}L_n$, where L_n denotes the integral of $|D_n(x)|$ over $(0, \pi)$. Now it is not difficult to prove that $L_n \sim \log n$ (see Ch. VIII). Hence

(ii) *Let $s_n(x)$ denote the partial sums of the series* 5.11(1a). *If $a_n \to 0$ and $\{a_n\}$ is quasi-convex, the relation $\mathfrak{M}[f-s_n]\to 0$ holds if and only if $a_n = o(1/\log n)$.*

If $a_n \log n \to \infty$, e. g. if $a_n = (\log n)^{-1/2}$, $n>1$, then $\mathfrak{M}[f-s_n]\to\infty$, $\mathfrak{M}[s_n]\to\infty$. The series

$$\sum_{n=2}^{\infty}\frac{\cos nx}{\log n}, \tag{2}$$

which plays an important part in some problems, is a limiting case, since here the sequence $\mathfrak{M}[f-s_n]$ is bounded and yet it does not tend to 0.

To complete the proof of (i), we observe that the series 5.11(1a) is certainly $\mathfrak{S}[f]$ if $\mathfrak{M}[f-s_n]\to 0$ (and in particular if $a_n \log n \to 0$). When this condition is not satisfied we must proceed otherwise and two ways are open for us. The first of them consists in proving that $\mathfrak{M}[f-\sigma_n]\to 0$ as $n\to\infty$, or that $\mathfrak{M}[f(x)-f(r,x)]\to 0$ as $r\to 1$, where $\sigma_n(x)$ and $f(r,x)$ denote respectively Fejér's and Abel's means of the series considered. We prefer to base the proof of (i) on the following theorem, which will be established in Chapter XI: *If a trigonometrical series converges, except at one point, to an integrable function f, the series is $\mathfrak{S}[f]$.*

Remarks. (a) Given an arbitrary sequence of positive numbers $\varepsilon_n \to 0$, we can easily construct, e. g. geometrically, a convex sequence $\{a_n\}$ such that $a_n \geqslant \varepsilon_n$, $a_n\to 0$. Thus *there exist Fourier series with coefficients tending to* 0 *arbitrarily slowly* (see also § 2.9.2).

(b) If a_n, b_n are the Fourier coefficients of an integrable function, the series $\Sigma\, b_n/n$ converges (§ 2.621). The example of the Fourier series (2) shows that the series $\Sigma\, a_n/n$ may be divergent.

5.121. In the preceding section we proved that, if $a_n \to 0$, $\Delta^2 a_n \geqslant 0$, the series 5.11(1a) is a Fourier series. We will now show that the condition $\Delta^2 a_n \geqslant 0$ cannot be replaced by $\Delta a_n \geqslant 0$. More precisely, *there exists a cosine series with coefficients monotonicaly decreasing to* 0 *and yet the sum* $f(x)$ *of this series is not integrable* ¹). In fact, let us suppose that there exists a sequence of integers $0 = \lambda_1 < \lambda_2 < \dots$ such that a_k is constant for $\lambda_n < k \leqslant \lambda_{n+1}$, $n = 1, 2, \dots$ Making Abel's transformation, we obtain for $f(x)$ the formula

$$(1) \qquad f(x) = \sum_{k=0}^{\infty} \Delta a_k D_k(x) = \sum_{n=1}^{\infty} \alpha_n D_{\lambda_n}(x),$$

where $\alpha_n = \Delta a_{\lambda_n}$. We require the following two inequalities

$$(2) \qquad \int_{1/n}^{\pi} |D_n(x)|\, dx > C \log n, \quad L_n = \int_0^{\pi} |D_n(x)|\, dx \leqslant C_1 \log n, \quad n = 2, 3, \dots,$$

where C and C_1 are positive constants. The second inequality is a corollary of the relation $L_n \sim \log n$, which will be proved in Chapter VIII. On the other hand, since $D_n(x) = O(n)$, the integral of $|D_n(x)|$ over $(0, 1/n)$ is $O(1)$, and the first inequality (2) is also a corollary of the relation $L_n \sim \log n$. From (1), (2), and the inequality $|D_n(x)| \leqslant 2/x$, $0 < x \leqslant \pi$, we see that

$$(3) \qquad \int_{1/\lambda_\nu}^{\pi} |f|\, dx \geqslant C \alpha_\nu \log \lambda_\nu - C_1 \sum_{n=1}^{\nu-1} \alpha_n \log \lambda_n - 2 \log(\pi \lambda_\nu) \sum_{n=\nu+1}^{\infty} \alpha_n.$$

Putting $\alpha_n = 1/n!$, $\lambda_n = 2^{(n!)^2}$, and arguing as in § 4.23, we obtain that the left-hand side of (3) is unbounded as $\nu \to \infty$.

5.13 ²). Next we shall consider the partial sums $\bar{s}_n(x)$ of the series 5.11(1b) with coefficients monotonically tending to 0. Let $\bar{D}_n(x) = \sin x + \dots + \sin nx = [\cos \frac{1}{2} x - \cos (n + \frac{1}{2}) x]/2 \sin \frac{1}{2} x$, $\tilde{D}_n(x) = [1 - \cos (n + \frac{1}{2}) x]/2 \sin \frac{1}{2} x \geqslant 0$, $0 \leqslant x \leqslant \pi$. We have

$$(1) \qquad \bar{s}_n(x) = \sum_{m=1}^{n} \Delta a_m \bar{D}_m(x) + a_{n+1} \bar{D}_n(x) \to \sum_{m=1}^{\infty} \Delta a_m \bar{D}_m(x) = \bar{f}(x).$$

Substituting $\tilde{D}_m$ for $\bar{D}_m$ in the last series we obtain a function $\tilde{f}(x)$ differing from $\bar{f}(x)$ by $\frac{1}{2} a_1 \operatorname{tg} {}^1/_4 x$. The series defining $\tilde{f}(x)$ has non-negative terms and, since the integrals of $\tilde{D}_n$ over $(0, \pi)$ are exactly of order $\log n$ (§ 2.631), we conclude that $\tilde{f}(x)$, and

---

¹) Szidon [1].

²) Young [9], Szidon [1], Hille and Tamarkin [1₂].

therefore $\bar{f}(x)$, is integrable if and only if the series with terms $\Delta a_n \cdot \log n$ converges.

As in § 5.12, we see that $\mathfrak{M}[\bar{f} - \bar{s}_n] \to 0$, provided that $\Delta a_2 \log 2 + \Delta a_3 \log 3 + \ldots < \infty$. (Observe that $a_n \log n \leqslant \Delta a_n \log n + \Delta a_{n+1} \log (n+1) + \ldots = o(1)$).

Since $a_n \to 0$, a simple calculation shows that

$$2\bar{f}(x) \sin x = a_1 + a_2 \cos x + \sum_{m=2}^{\infty} (a_{m+1} - a_{m-1}) \cos mx.$$

The series on the right, which is uniformly convergent, is $\mathfrak{S}[2\bar{f} \sin x]$. Writing the Fourier formulae for the coefficients $a_1, a_2, a_3 - a_1, \ldots$ of the last series, we obtain, by addition of some of these formulae, that

$$a_n = \frac{2}{\pi} \int_0^{\pi} \bar{f}(x) \sin nx \, dx, \quad n = 1, 2, \ldots \tag{2}$$

Collecting the results we may enounce the following theorem.

If $a_n \geqslant a_{n+1} \to 0$, the sum $\bar{f}(x)$ of the series 5.11(1b) *is bounded below in the interval* $(0, \pi)$, *and we have the formula* (2), *where $\bar{f} \sin nx$ is continuous*[1]. *A necessary and sufficient condition for the integrability of $\bar{f}$ is the convergence of the series $\Delta a_2 \log 2 + \Delta a_3 \log 3 + \ldots$ If this condition is satisfied then $\mathfrak{M}[\bar{f} - \bar{s}_n] \to 0$.*

If $a_n \geqslant a_{n+1} \to 0$, the convergence of the series $\Delta a_2 \log 2 + \ldots$ implies that of $a_1 + {}^1/_2\, a_2 + {}^1/_3\, a_3 + \ldots$ and vice versa. The first part of this proposition follows from Abel's transformation, if we observe that $\log n - \log (n-1) \simeq 1/n$. For the second part we must use the fact that, if $a_1 + \frac{1}{2} a_2 + \ldots < \infty$, then $a_1 + a_2/2 + \ldots + a_n/n \geqslant a_n(1 + \ldots + 1/n)$ and so $a_n = O(1/\log n)$.

5.2. Approximate expressions for certain series[2]. It is important in some cases to know the behaviour of the series 5.11(1) in the neighbourhood of the point $x = 0$, and we intend to give approximate expressions for their sums, which we shall denote by $f(x)$, $\bar{f}(x)$ respectively.

5.21. We suppose that the coefficients a_n in 5.11(1b) form a sequence decreasing monotonically to 0 and convex. Put $x_p = \pi/2p$.

---

[1] The continuity of $\bar{f} \sin nx$ follows from that of $\bar{f} \sin x$.

[2] Salem [1]. Less precise results had been obtained previously by Young [3].

A simple computation shows that $\bar{f}(x_p) = b_1 \sin x_p + b_2 \sin 2x_p + \dots + b_p \sin p\, x_p$, where $b_j = b_j^{(p)}$, $j=1, 2, \dots, p-1$, may be written in either of the forms

$$b_j = a_j + (a_{2p-j} - a_{2p+j}) - (a_{4p-j} - a_{4p+j}) + \dots ,$$

$$b_j = (a_j + a_{2p-j}) - (a_{2p+j} + a_{4p-j}) + (a_{4p+j} + a_{6p-j}) - \dots ,$$

and $b_p = b_p^{(p)} = a_p - a_{3p} + a_{5p} \dots$

Since a_n and Δa_n decrease, the expressions in brackets also decrease, and we find that $a_j \leqslant b_j \leqslant a_j + a_{2p-j}$, i. e. $a_j \leqslant b_j \leqslant 2a_j$, $j = 1, 2, \dots p-1$. Observing that $u \geqslant \sin u \geqslant 2u/\pi$ for $0 \leqslant u \leqslant \pi/2$, we find that the ratio of $\bar{f}(x_p) - b_p$ to $[a_1 + 2a_2 + \dots + (p-1)\, a_{p-1}]/p$ is contained betwen 1 and π.

To find a simpler expression for $\bar{f}(x_p)$ we shall make an additional asumption about $\{a_n\}$, viz. that *na_n is non-decreasing*. To elucidate this hypothesis we observe that in all the series 5.11(1) that occur in practice and have coefficients steadily decreasing to 0, na_n is monotonic, at least for n sufficiently large. Moreover, if na_n is non-increasing, the function $\bar{f}(x)$ is continuous, or has a simple discontinuity, at the point $x = 0$ (§ 5.11).

If a_n is non-increasing and na_n non-decreasing, then $[a_1 + 2a_2 + \dots + (p-1)\, a_{p-1}]/p$ is contained between $\frac{1}{2}(p-1)\, a_{p-1}$ and pa_p or, à fortiori, between $\frac{1}{2} pa_p - \frac{1}{2} a_p$ and pa_p. Since pa_p is bounded below by a positive number, and $0 < b_p < a_p$ we find, finally, that $\bar{f}(x_p) \sim pa_p$. To find a formula for an arbitrary $x \to 0$ we require the following lemma.

If x'_p is an arbitrary point in the interval $\pi/2p \leqslant x \leqslant \pi/2(p-1)$, then $\bar{f}(x'_p) - \bar{f}(x_p) = o(pa_p)$ as $p \to \infty$.

In the formula 5.13(1) we break up the sum defining $\bar{f}(x)$, into two parts $P(x)$ and $Q(x)$, P consisting of terms with indices $\leqslant rp$, where r is a fixed but large integer. Since $|\tilde{D}'_k(x)| \leqslant \leqslant 1 + 2 + \dots + k \leqslant k^2$, we find, by the mean-value theorem, that

$$|P(x'_p) - P(x_p)| \leqslant (x'_p - x_p)\, [\Delta a_1 \cdot 1^2 + \dots + \Delta a_{pr} \cdot (pr)^2] \to 0,$$

since $(x_p - x'_p) \leqslant \pi/2p\,(p-1)$, $k\Delta a_k \to 0$, and so $k^2 \Delta a_k = o(k)$.

Remembering that $\tilde{D}(x) = [\frac{1}{2} \operatorname{ctg} \frac{1}{2} x - \cos(n + \frac{1}{2})\, x]/2 \sin \frac{1}{2} x$, we put accordingly $Q = Q_1 + Q_2$, where $Q_1 = \frac{1}{2} \operatorname{ctg} \frac{1}{2} x \cdot (\Delta a_{pr+1} + \dots) = a_{pr+1} \cdot \frac{1}{2} \operatorname{ctg} \frac{1}{2} x$. It is easy to see that $Q_1(x_p) - Q_1(x'_p) = o(1)$ as $p \to \infty$. Since Δa_n is non-increasing, we find that $|Q_2(x_p)|$ and

$|Q_2(x_p')|$ do not exceed $Cp^2 \Delta a_{pr}$, where C is an absolute constant (§ 1.22). Now the inequality $na_n \leqslant (n+1)\, a_{n+1}$ involves $n\,\Delta a_n \leqslant a_n$ and therefore

$$Cp^2 \Delta a_{pr} = C\,(p/r)\, pr\, \Delta a_{pr} \leqslant C\,(p/r)\, a_{pr} \leqslant (C/r)\, pa_p \leqslant \varepsilon\, pa_p,$$

if r is sufficiently large. Collecting our inequalities together, we obtain ultimately $|f(x_p') - f(x_p)| \leqslant |P(x_p) - P(x_p')| + |Q_1(x_p') - Q(x_p)| + |Q_2(x_p)| + |Q_2(x_p')| \leqslant o(1) + o(1) + 2\,\varepsilon\, pa_p < 3\,\varepsilon\, pa_p$ for p large. Since ε is arbitrary, the lemma follows.

From what we have proved it follows that $\bar{f}(x) \sim pa_p$, *where the integer* p *is defined by the condition* $\pi/2p \leqslant x < \pi/2\,(p-1)$. It is however preferable to state this result in a slightly different form. We may always suppose that $a_n = a(n)$, where $a(x)$ is a convex and decreasing function of x. Indeed in most cases a_n is just given as $a(n)$, but even if it be not so we can, for example, define $a(x)$ by the condition of continuity and that of being linear in every interval $(n, n+1)$.

5.211. *Let* $a(x)$, $x \geqslant 0$, *be a function decreasing to* 0, *convex, and such that* $na(n)$ *is non-decreasing. If* $a(n) = a_n$, *the sum of the series* 5.11(1b) *satisfies the relation* $\bar{f}(x) \sim x^{-1}\, a(x^{-1})$ *as* $x \to 0$.

In fact, if $p = p_x = [\pi/2x] + 1$, then $\pi/2p \leqslant x \leqslant \pi/2\,(p-1)$ and, by the previous result, $\bar{f}(x) \sim pa(p) \sim x^{-1}\, a(p)$. It remains only to show that $a(p) \sim a(x^{-1})$. For small x we have $x^{-1} \leqslant p \leqslant 2x^{-1}$. From the first inequality we see that $a(x^{-1}) \geqslant a(p)$. From the second, assuming p even, we deduce that $a(x^{-1}) \leqslant a(\frac{1}{2}p) = (2/p)\,(p/2) a(\frac{1}{2}p) \leqslant (2/p)\, pa(p) = 2a(p)$. Using the inequality $p + 1 \leqslant 2x^{-1}$, which is true for small x, we find that $a(x^{-1}) \leqslant 2a(p)$ for p odd, and so in any case $a(p) \leqslant a(x^{-1}) \leqslant 2a(p)$. This completes the proof.

5.22. Supposing the sequence $a_0, a_1, \ldots$ convex and decreasing to 0, we find for the series 5.11(1a) the estimates

(1) $$f(x_p) \leqslant \tfrac{1}{2} a_0 + \sum_{k=1}^{p-1} (a_k - a_{2p-k}) \cos kx_p,$$

(2) $$f(x_p) \geqslant \tfrac{1}{2} a_0 + \sum_{k=1}^{p-1} [(a_k - a_{2p-k}) - (a_{2p+k} - a_{4p-k})] \cos kx_p.$$

Replacing in (1) a_0 by $(a_0 - a_1) + \ldots + (a_{2p-1} - a_{2p}) + a_{2p}$, and $a_k - a_{2p-k}$ by $(a_k - a_{k+1}) + \ldots + (a_{2p-k-1} - a_{2p-k})$, we find that

(3) $$f(x_p) \leqslant 2\,[\tfrac{1}{2}\Delta a_0 + \sum_{k=1}^{p-1} \Delta a_k\, D_k(x_p)] + \tfrac{1}{2} a_{2p},$$

where D_k denotes Dirichlet's kernel. To obtain a lower bound for $f(x)$, we shall make an additional hypothesis concerning $\{a_k\}$, viz. that $k(a_k - a_{k+1})$ is a non-increasing function of k (from the convexity of $\{a_k\}$ we only have $k\,\Delta a_k \to 0$). From this assumption we deduce that $(a_{2p+k} - a_{4p-k}) \leqslant \frac{1}{2}(a_k - a_{2p-k})$, $k = 1, 2, \ldots, p-1$, and therefore, using (2), that

$$(4)\qquad f(x_p) \geqslant \tfrac{1}{2}\,[\tfrac{1}{2}\Delta a_0 + \sum_{k=1}^{p-1} \Delta a_k D_k(x_p)].$$

It is natural to suppose that $a_1 + a_2 + \ldots = \infty$. Thence it follows that $\Delta a_1 + 2\Delta a_2 + \ldots + (p-1)\Delta a_{p-1} = (a_1 - a_p) + (a_2 - a_p) + \ldots + (a_{p-1} - a_p) \to \infty$, and from (3), (4) we conclude that $f(x_p) \sim \Delta a_1 + 2\Delta a_2 + \ldots + (p-1)\,\Delta a_{p-1} \sim \Delta a_1 + 2\Delta a_2 + \ldots + p\,\Delta a_p$.

Now let x'_p be any point in the interval $(\pi/2p, \pi/2(p-1))$. We find, as previously, that $|f(x_p) - f(x'_p)| \leqslant o(1) + o(p^2 \Delta a_p)$. This, together with the inequality $p^2 \Delta a_p \leqslant \Delta a_1 + 2\Delta a_2 + \ldots + p\,\Delta a_p$, yields the final result: $f(x) \sim \Delta a_1 + 2\Delta a_2 + \ldots + p\,\Delta a_p$, *where p satisfies the condition* $\pi/2p \leqslant x < \pi/2(p-1)$.

5.221. *If* $a(x)$, $x \geqslant 0$, *is a positive and convex function, tending to* 0, *then for the sum* $f(x)$ *of the series* 5.11(1a), *with* $a_n = a(n)$, $n(a_n - a_{n+1})$ *non-increasing, and* $a_0 + a_1 + \ldots = \infty$, *we have the formulae*

$$(1)\qquad f(x) \sim \int_1^{1/x} t\,[a(t) - a(t+1)]\,dt \sim \int_0^{1/x} t\,|a'(t)|\,dt.$$

To prove the first formula let us put $g_k = \Delta a_1 + 2\,\Delta a_2 + \ldots + k\,\Delta a_k$, and let $F(x)$ be the first integral in (1). We have to prove that $F(x) \sim g_p$, where $p > 1/x$ has the same meaning as in § 5.211. Let q be the largest integer $\leqslant 1/x$. Since $a(t)$ is convex, $a(t) - a(t+1)$ is non-increasing, and it is easy to see that $F(x) \geqslant g_q - a_1$. Similarly we find that $F(x) \leqslant F(1/p) \leqslant g_p + a_1$. From the inequalities $g_q \leqslant g_p = g_q + (g_p - g_q) \leqslant g_q + (p-q)\,q\Delta a_q = g_q + O(q^2 \Delta a_q) = g_q + O(g_q) = O(g_q)$, we see that $g_p \sim g_q$, and so $F(x) \sim g_p$.

Let $H(x)$ be the second integral in (1). To prove the second formula in (1) it is sufficient to show that $F(x) \sim H(x)$. This, and even a stronger result, viz. $F(x) \simeq H(x)$, follows from the inequalities $|a'(t)| \geqslant |a(t) - a(t-1)| \geqslant |a'(t+1)|$. The details of the proof may be left to the reader.

In the above proof we assumed tacitly that $a'(t)$ exists. The existence of $a'(t)$ follows, except for a set of t which is at most enumerable and has no influence upon the integral, from the mere convexity of $a(t)$ (§ 4.141). Let us assume now that $a''(x)$ exists. The inequality $n \Delta a_n \geqslant (n+1) \Delta a_{n+1}$ will certainly be satisfied if only (*) $a'(x) + (x-1) a''(x) \geqslant 0$. This test may be proved as follows. Let $\alpha(x) = x[a(x) - a(x+1)]$; then $\alpha'(x) = a(x) - a(x+1) + x[a'(x) - a'(x+1)]$. By the mean value theorem we shall have $\alpha'(x) \leqslant 0$ provided that $a'(x+\theta)/a''(x+\theta) + x \geqslant 0$, where θ is a number contained between 0 and 1, and the latter inequality is a consequence of (*). Of course it is sufficient for (*) to be satisfied for x large.

Examples. If $a_n = n^{-\alpha}$, $0 < \alpha < 1$, $n \geqslant 1$, then $f(x)$ and $\bar{f}(x)$ are of order $x^{\alpha-1}$ as $x \to +0$. If $a_n = 1/\log n$, $n \geqslant 2$, then $f(x) \sim 1/x(\log x)^2$, $\bar{f}(x) \sim 1/x|\log x|$, as $x \to 0$. In particular the series

(2) $$\sum_{n=2}^{\infty} \frac{\sin nx}{\log n},$$

which converges everywhere, is not a Fourier series. This follows also from the fact that the series (2) integrated term by term diverges at the point 0 (§ 2.621).

5.3. A power series. We shall now consider the power series

(1) $$\sum_{n=1}^{\infty} e^{icn \log n} \frac{z^n}{n^{1/2+\alpha}},$$

where α and $c \neq 0$ are real constants, $z = e^{ix}$, $0 \leqslant x \leqslant 2\pi$. The series (1), which was first studied by Hardy and Littlewood, possesses many interesting properties.

If $0 < \alpha < 1$, *the series* (1) *converges uniformly in the interval* $0 \leqslant x \leqslant 2\pi$ *to a function* $\varphi_\alpha(x) \in \text{Lip}\,\alpha$ [1]).

The theorem is a corollary of certain lemmas, which are interesting in themselves and have wider applications.

5.31. van der Corput's lemmas. Given a real function $f(u)$ and numbers $a < b$, we put

$$F(u) = e^{2\pi i f(u)}, \quad I(F; a, b) = \int_a^b F(u)\,du, \quad S(F; a, b) = \sum_{a < n \leqslant b} F(n).$$

---

[1]) Hardy and Littlewood [9]. Following Hille [1], we base our proof on van der Corput's lemmas. See van der Corput [1].

(i) *If* $f(u)$, $a \leqslant x \leqslant b$, *has an increasing derivative* $f'(u)$, *and if* $f''(u) \geqslant \rho > 0$, *then* $|I(F; a, b)| < 4\rho^{-1/2}$.

Suppose that there exists a $\lambda > 0$ such that $f'(u) \geqslant \lambda$, or $f'(u) \leqslant -\lambda$, throughout (a, b). Since $2\pi i F(u)\, du = dF(u)/f'(u)$, an application of the second mean-value theorem to the real and imaginary parts of $I(F; a, b)$ shows that $|I| \leqslant 2/\pi\lambda < 1/\lambda$.

Assuming that the conditions of the lemma are satisfied, suppose for the moment that $f'(u)$ is of constant sign, say $f' \geqslant 0$, in (a, b). If $a < c < b$, then $f'(u) \geqslant (c-a)\rho$ in the interval $c \leqslant u \leqslant b$. Therefore $|I(F; a, b)| \leqslant |I(F; a, c)| + |I(F; c, b)| < (c-a) + 1/(c-a)\rho$. Choosing c so as to make the last expression a minimum, we find that $|I(F; a, b)| < 2\rho^{-1/2}$. In the general case (a, b) is a sum of two intervals in each of which $f'(u)$ is of constant sign, and the result follows by the addition of the inequalities for these intervals.

(ii) *Let* $D(F; a, b) = I(F; a, b) - S(F; a, b)$. *If* $f'(u)$ *is monotonic and* $|f'(u)| \leqslant \frac{1}{2}$, *then* $|D(F; a, b)| \leqslant A$, *where* A *is an absolute constant.*

Suppose first that a and b are not integers. S may be written as the Stieltjes integral of $F(u)\, d\xi(u)$ over (a, b), where $\xi(u)$ is any function which is constant in the intervals $n < u < n+1$ and has jumps equal to 1 at the points n. It will be convenient to put $\xi(u) = [u] + \frac{1}{2}$ for $u \neq 0, \pm 1, \ldots$, $2\xi(u) = \xi(u+0) + \xi(u-0)$. It follows that

$$D(F; a, b) = \int_a^b F(u)\, d\chi(u), \quad \text{where } \chi(u) = u - [u] - \tfrac{1}{2}.$$

The function $\chi(u)$ is of period 1. Integrating by parts, we find that $D(F; a, b) = -I(F'\chi; a, b) + R$, where $|R| \leqslant 1$. The terms of $\mathfrak{S}[\chi]$ are $-\sin 2\pi nu/\pi n$ and the partial sums are uniformly bounded. Multiplying $\mathfrak{S}[\chi]$ by F' and integrating over (a, b), we see that $D(F; a, b) - R$ is equal to the sum of the expressions

$$(1) \qquad \frac{1}{2\pi i n}\left[\int_a^b \frac{f'(u)}{f'(u)+n}\, de^{2\pi i [f(u)+nu]} - \int_a^b \frac{f'(u)}{f'(u)-n}\, de^{2\pi i [f(u)-nu]}\right]$$

for $n = 1, 2, \ldots$. The factors $f'/(f' \pm n)$ are monotonic and of constant sign. The second mean-value theorem shows that (1) does not exceed $2/\pi n(n-\frac{1}{2})$ in absolute value, and so the series of expressions (1) converges absolutely. This completes the proof in the case when a and b are not integers. If a or b, or both, are integers, it is sufficient to observe that $D(F; a, b)$ differs from $\lim_{\varepsilon\to+0} D(F; a+\varepsilon, b-\varepsilon)$ by at most 1.

(iii) *Under the conditions of* (i) *we have*

$$|S(F; a, b)| \leqslant [f'(b) - f'(a) + 2]\,(4\rho^{-1/2} + A).$$

Put $\beta_p = p - \frac{1}{2}$, $p = 0, \pm 1, \dots$, and let $\beta_p = f'(\alpha_p)$, $F_p(u) = \exp 2\pi i[f(u) - pu]$. It is obvious that $|f'(u) - p| \leqslant \frac{1}{2}$ in the interval (α_p, α_{p+1}). Let $\alpha_r, \alpha_{r+1}, \dots, \alpha_{r+s}$ be all the points α, if such exist, belonging to the interval $a \leqslant u \leqslant b$. Using (i) and (ii), we see that the expressions $S(F; \alpha_p, \alpha_{p+1}) = S(F_p; \alpha_p, \alpha_{p+1}) = I(F_p; \alpha_p, \alpha_{p+1}) - D(F_p; \alpha_p, \alpha_{p+1})$ do not exceed $4\rho^{-1/2} + A$ in absolute value. The same may be said of $S(F; a, \alpha_r)$ and $S(F; \alpha_{r+s}, b)$. Since $S(F; a, b)$ is a sum of analogous expressions formed for the intervals (a, α_r), $(\alpha_r, \alpha_{r+1}), \dots, (\alpha_{r+s}, b)$, the number of which is $s + 2 = f'(\alpha_{r+s}) - f'(\alpha_r) + 2 \leqslant f'(b) - f'(a) + 2$, the result follows.

5.32. *The partial sums $s_N(x)$ of the series* 5.3(1) *with* $\alpha = -\frac{1}{2}$, *are $O(N^{1/2})$ uniformly in x.*

The function $f(u) = (2\pi)^{-1}(cu\log u + ux)$ has an increasing derivative. If $\nu \geqslant 0$ is an integer, $a = 2^\nu$, $b = 2^{\nu+1}$, we conclude from § 5.31(iii) that $|S(F; a, b)| \leqslant C2^{\nu/2}$ with C depending only on c. The same is true if $2^\nu = a < b < 2^{\nu+1}$. Let $2^n < N \leqslant 2^{n+1}$. Then $|s_N(x)| \leqslant 1 + |S(F; 1, 2)| + |S(F; 2, 4) + \dots + |S(F; 2^n, N)| \leqslant 1 + C\{2^{1/2} + \dots + 2^{n/2}\} \leqslant C_1 2^{n/2} \leqslant C_1 N^{1/2}$, with C_1 depending only on c.

We can now easily prove Theorem 5.3. Using Abel's transformation we obtain for the N-th partial sum of the series 5.3(1) the expression

$$\sum_{\nu=1}^{N-1} s_\nu(x)\,\Delta\nu^{-1/2-\alpha} + s_N(x)\,N^{-1/2-\alpha}. \tag{1}$$

Since $\Delta\nu^{-1/2-\alpha} = O(\nu^{-3/2-\alpha})$, we conclude from (1) and from the relation $s_\nu(x) = O(\nu^{1/2})$ that the partial sums of 5.3(1) are (a) uniformly convergent if $\alpha > 0$, (b) uniformly $O(\log N)$ if $\alpha = 0$, (c) uniformly $O(N^{-\alpha})$ if $\alpha < 0$. Take $0 < \alpha < 1$. Making $N \to \infty$ in (1) we obtain

$$\varphi_\alpha(x+h) - \varphi_\alpha(x) = \sum_{\nu=1}^{\infty}\{s_\nu(x+h) - s_\nu(x)\}\,\Delta\nu^{-1/2-\alpha} = \sum_{\nu=1}^{N} + \sum_{\nu=N+1}^{\infty} = P + Q,$$

where $h>0$, $N=[1/h]$. The terms in Q are $O(\nu^{1/2})\,\Delta\nu^{-1/2-\alpha}=O(\nu^{-1-\alpha})$, so that $Q=O(N^{-\alpha})=O(h^{\alpha})$. On the other hand, since $s'_\nu(x)$, apart from a constant factor, is the partial sum of the series 5.3(1) with $\alpha=-3/2$, we have (see case (c) above) that $s'_\nu(x)=O(\nu^{3/2})$. Applying the mean-value theorem to $s_\nu(x+h)-s_\nu(x)$, we find that the terms of P are $O(h\nu^{3/2})\,\Delta\nu^{-3/2-\alpha}=O(h\nu^{-\alpha})$, and so $P=O(hN^{1-\alpha})=$ $=O(h^{\alpha})$. Therefore $|\varphi_\alpha(x+h)-\varphi_\alpha(x)|\leqslant|P|+|Q|=O(h^{\alpha})$ and the theorem follows.

5.33. Theorem 5.3 ceases to be true when $\alpha=0$ (and so when $\alpha=1$). In this case much more can be said: if $\alpha=0$, the series 5.3(1) is nowhere summable A, and, à fortiori, is not a Fourier series [1]). However, if $\beta>1$, $c\gtrless 0$, the series

(1) $$\sum_{n=2}^{\infty}\frac{e^{icn\log n}}{n^{1/2}(\log n)^{\beta}}\,z^n,\quad z=e^{ix},$$

converges uniformly for $0\leqslant x\leqslant 2\pi$. For the proof we replace $\Delta\nu^{-1/2-\alpha}$ by $\Delta\nu^{-1/2}\log^{-\beta}\nu=O(\nu^{-3/2}\log^{-\beta}\nu)$ [2]) in 5.32(1), $N^{-1/2-\alpha}$ by $N^{-1/2}\log^{-\beta}N$, and observe that the series with terms $O(\nu^{-1}\log^{-\beta}\nu)$ converges.

5.34. *There exists a continuous function $f(x)$ such that, if a_n, b_n are the Fourier coefficients of f, the series $\Sigma(|a_n|^{2-\varepsilon}+|b_n|^{2-\varepsilon})$ diverges for every $\varepsilon>0$* [3]). For, if $f(x)$ is the real, or imaginary, part of the function 5.33(1), where $\beta>1$, and $\rho_n^2=a_n^2+b_n^2$, $\rho_n\geqslant 0$, then $\Sigma\rho_n^{2-\varepsilon}=\infty$ for $\varepsilon>0$, and this is equivalent to our theorem.

5.4. Lacunary series. We now pass to the lacunary trigonometrical series, that is to series where the terms different from 0 are 'very sparse'. Such series may be written in the form

(1) $$\sum_{k=1}^{\infty}(a_k\cos n_k x+b_k\sin n_k x),$$

assuming, for simplicity, that the constant term also vanishes. When speaking on lacunary series, we shall suppose throughout

---

[1]) Hardy and Littlewood [9].

[2]) This inequality follows from the mean-value theorem applied to the difference $\alpha(n)-\alpha(n+1)$, where $\alpha(x)=x^{-1/2}\log^{-\beta}x$.

[3]) The first example of a continuous function having this property was given by Carleman [1].

that the indices n_k satisfy an inequality $n_{k+1}/n_k > \lambda > 1$, i. e. increase at least as rapidly as a geometrical progression with ratio greater than 1.

Given a lacunary series (1) consider the sum

$$\sum_{k=1}^{\infty}(a_k^2+b_k^2). \tag{2}$$

In Chapter X we shall learn that, if (2) is finite, the series (1) converges almost everywhere. Here we shall prove the converse. *If the series* (1) *converges in a set of positive measure, the series* (2) *converges.* We shall prove even a more general theorem. Let T^* be any linear method of summation satisfying the two first conditions of Toeplitz (§ 3.1); the third condition need not be satisfied. All methods of summation used in Analysis are T^*-methods.

If a series of the form (1) *is summable* T^* *in a set* E *of positive measure, the series* (2) *converges*[1]).

It must be observed that, when we speak of the summability of the series (1), we understand that the vacant terms are replaced by zeros. Consequently, the q-th partial sum $s_q(x)$ of (1) consists of the terms $a_k \cos n_k x + b_k \sin n_k x$ with $n_k \leqslant q$. If β_{pq} denotes an element of the matrix T^* considered, the hypothesis of the last theorem may be stated as follows: for every $x \in E$ the series

$$\sum_{q=0}^{\infty}\beta_{pq}\, s_q(x) = \sigma_p(x), \qquad p = 0, 1, 2, \ldots, \tag{3}$$

converge, and $\lim \sigma_p(x)$ exists and is finite.

To avoid unnecessary complications we begin by the case when each line of the matrix $\{\beta_{pq}\}$ possesses only a finite number of terms different from 0. It will be convenient to consider the series (1) in the complex form, putting $2c_k = a_k - ib_k$, $c_{-k} = \bar{c}_k$, $c_0 = 0$, $n_{-k} = -n_k$, $k = 1, 2, \ldots$ Let, moreover, $\beta_{p,q} + \beta_{p,q+1} + \ldots = R_p(q)$. It is easy to see that

$$\sigma_p(x) = \sum_{k=-\infty}^{+\infty} c_k e^{in_k x} R_p(|n_k|),$$

the sum on the right being in reality finite. Since $\{\sigma_p(x)\}$ converges in E, we can find a subset $\mathcal{E}$ of E, $|\mathcal{E}| > 0$, and a number

[1]) Zygmund [5]; see also Kolmogoroff [2].

M, such that $|\sigma_p(x)| \leqslant M$ for $p = 0, 1, \dots, x \in \mathcal{E}$. In fact, we have $E = E_1 + E_2 + \dots$, where E_n is the set of x such that $|\sigma_p(x)| \leqslant n$ for $p = 0, 1, 2, \dots$ At least one of the sets E_i, say E_M, is of positive measure and may be taken for $\mathcal{E}$. It follows that

$$(4)\quad \begin{aligned} M^2 |\mathcal{E}| \geqslant \int_{\mathcal{E}} \sigma_p^2(x)\,dx = |\mathcal{E}| \sum_{k=-\infty}^{+\infty} |c_k|^2 R_p^2(|n_k|) + \\ + \sum_{\substack{j,k=-\infty \\ j \neq k}}^{+\infty} c_j \bar{c}_k R_p(|n_j|) R_p(|n_k|) \int_{\mathcal{E}} e^{i(n_j - n_k)x}\,dx. \end{aligned}$$

Let us denote the last integral by $2\pi b_{j,k}$. The numbers $b_{j,k}$ are the complex Fourier coefficients of a function $\chi(x)$ which is equal to 1 in $\mathcal{E}$ and to 0 elsewhere. Applying Schwarz's inequality to the second sum on the right, we see that it does not exceed

$$(5)\quad \begin{aligned} 2\pi \Big\{ \sum_{j,k=-\infty}^{+\infty} |c_j|^2 |c_k|^2 R_p^2(|n_j|) R_p^2(|n_k|) \Big\}^{1/2} \Big\{ \sum_{\substack{j,k=-\infty \\ j \neq k}}^{+\infty} |b_{j,k}|^2 \Big\}^{1/2} = \\ = 2\pi \sum_{k=-\infty}^{+\infty} |c_k|^2 R_p^2(|n_k|) \cdot \Big\{ \sum_{\substack{j,k=-\infty \\ j \neq k}}^{+\infty} |b_{j,k}|^2 \Big\}^{1/2} \end{aligned}$$

in absolute value.

From the condition $n_{k+1}/n_k > \lambda > 1$ it follows that a number $\Delta = \Delta(\lambda)$ exists such that every integer m can be represented no more than Δ times in the form $n_j \pm n_k$, $j > 0$, $k > 0$. In fact, assume that $m = n_j + n_k$, $j \geqslant k$. Then $m > n_j \geqslant m/2$, and the number of n_j satisfying this inequality is less than the smallest integer y such that $\lambda^y > 2$. Similarly, if $m = n_j - n_k > 0$, then $n_j > m$. As $n_j/n_k > \lambda$, we have $n_j - n_j/\lambda < m$, i. e. $n_j < m\lambda/(\lambda - 1)$, and the number of n_j in the interval $(m, m\lambda/(\lambda - 1))$ is also bounded. We add that the property of $\{n_j\}$ just established is the only thing which we use in the proof, and that it may sometimes hold even if $n_{j+1}/n_j \to 1$ as $j \to \infty$.

Now it is not difficult to see that the last factor on the right in (5) does not exceed $\{\Delta(\dots + |\gamma_{-1}|^2 + |\gamma_0|^2 + |\gamma_1|^2 + \dots)\}^{1/2} < \infty$, where γ_m denote the complex Fourier coefficients of χ. Thence, for ν sufficiently large, we have

$$(6)\quad 2\pi \Big(\sum_{\substack{|j|,|k| > \nu \\ j \neq k}} |b_{j,k}|^2 \Big)^{1/2} < \tfrac{1}{2} |\mathcal{E}|.$$

In the series (1) we may omit the terms $a_k \cos n_k x + b_k \sin n_k x$ for $1 \leqslant k \leqslant \nu$, replacing them by zeros. It does no damage to the summability T^* of the series considered and can only change the value of M. Assuming the inequality (6), we deduce from (4) and (5) that

$$M^2 |\mathcal{E}| \geqslant \tfrac{1}{2} |\mathcal{E}| \sum_{k=-\infty}^{+\infty} |c_k|^2 R_p^2(|n_k|).$$

Let $K > 0$ be any fixed integer. Since $\lim_p R_p(|n_k|) = 1$, $k = 1, 2, \ldots$, we conclude that

$$\sum_{k=-K}^{K} |c_k|^2 R_p^2(|n_k|) \leqslant 2M^2, \qquad \sum_{k=-K}^{K} |c_k|^2 \leqslant 2M^2,$$

and, since the last inequality holds for any K, the convergence of (2) follows.

To remove the condition imposed upon $\{\beta_{pq}\}$ we proceed as follows. Let $\sigma_p^*(x)$ be an expression analogous to $\sigma_p(x)$ (see (3)), except that the upper limit of summation in the sum defining σ_p^* is not ∞ but a number $Q = Q(p)$. We take Q very large, so as to satisfy the two following conditions (i) $|\sigma_p(x) - \sigma_p^*(x)| \leqslant 1/p$ for $x \in E - E^p$, where the set E^p is of measure $\leqslant 2^{-p-1} |E|$, $p = 1, 2, \ldots$ (ii) $\lim_p (\beta_{p0} + \beta_{p1} + \ldots + \beta_{pQ}) = 1$. Putting $E^* = E^1 + E^2 + \ldots$, so that $|E^*| \leqslant \frac{1}{2} |E|$, we see that in the set $E - E^*$ of positive measure the expressions $\sigma_p^*(x)$ tend to a finite limit. But condition (ii) ensures that the σ_p^* are also T^*-means, corresponding to a matrix with only a finite number of terms different from 0 in each row, and, in virtue of the special case already dealt with, the theorem is established completely.

This theorem shows that, if the series (2) is infinite, the series (1) is practically non-summable by any method of summation. Considering, in particular, the method $(C, 1)$, we obtain: *if the series* (2) *diverges*, (1) *is not a Fourier series.*

5.5. Rademacher's series. Several properties of lacunary trigonometrical series are shared by Rademacher's series

$$(1) \qquad \sum_{k=0}^{\infty} c_k \varphi_k(t), \quad 0 \leqslant t \leqslant 1,$$

(§ 1.32). This is not surprising since Rademacher's functions form a lacunary subsequence of a complete orthogonal system (§ 1.8.5).

(i) *The series* (1) *converges almost everywhere if* $c_0^2+c_1^2+\dots<\infty$ [1]. (ii) *If* $c_0^2+c_1^2+\dots=\infty$, *the series* (1) *is almost everywhere non-summable by any method* T^* [2].

The proof of (ii) follows exactly the same line as that of Theorem 5.4 and may be left to the reader. We need only observe that the system of functions $\varphi_{j,k}(t)=\varphi_j(t)\,\varphi_k(t)$, $0\leqslant j<k$, $0\leqslant k<\infty$, is orthogonal and normal in (0, 1).

Under the hypothesis of (i), the series (1), whose partial sums we denote by $s_n(t)$, is the Fourier series of a function $f(t)\in L^2$ (§ 4.21) and moreover we have

$$\int_0^1 (f-s_n)^2\,dt\to 0,\quad \int_0^1 |f-s_n|\,dt\to 0,\quad \int_a^b (s_n-f)\,dt\to 0,$$

where $0\leqslant a<b\leqslant 1$. The third relation, which holds uniformly in a, b, is a consequence of the second, and the second follows from the first by an application of Schwarz's inequality.

Let us denote by $F(t)$ the indefinite integral of $f(t)$, and by E, $|E|=1$, the set of points where $F'(t)$ exists and is finite. We have proved that, whatever the interval I, the integral of s_n over I tends to the corresponding integral of f. Therefore, the integral of s_n-s_{k-1} over I tends, as $n\to\infty$, to the integral of $f-s_{k-1}$. Let I be of the form $(l2^{-k}, (l+1)2^{-k})$, $l=0,1,\dots,2^k-1$. Since the integral of $\varphi_j(t)$, over I vanishes for $j\geqslant k$, the integral of $s_n(t)-s_{k-1}(t)$ over I is equal to 0, provided that $n\geqslant k$. Hence, if I is of the form $(l2^{-k}, (l+1)2^{-k})$, the integral of $f(t)$ over I is equal to the integral of $s_{k-1}(t)$ over I. Now let $t_0\neq p/2^q$, $t_0\in E$, and let $t_0\in I_k=(l2^{-k}, (l+1)2^{-k})$. Since $s_{k-1}(t)$ is constant over I_k, we have

$$s_{k-1}(t_0)=\frac{1}{|I_k|}\int_{I_k} s_{k-1}(t)\,dt=\frac{1}{|I_k|}\int_{I_k} f(t)\,dt\to F'(t_0)\ \text{as}\ k\to\infty.$$

5.51. (i) *If the series* 5.5(2) *is convergent, the sum* $f(t)$ *of the series* 5.5(1) *belongs to* L^q *for every* $q>0$ [3]. It is sufficient to prove the theorem for $q=2,4,6,\dots$ We shall show that

---

[1]) Rademacher [1], see also Paley and Zygmund [1], and Kolmogoroff [3], where a very simple proof is given.

[2]) Khintchine and Kolmogoroff [1] (for the case of convergence). Zygmund [5].

[3]) Khintchine [1], Paley and Zygmund [1].

$$(1) \qquad \int_0^1 f^{2k}(t)\,dt \leqslant M_k \left(\sum_{n=0}^{\infty} c_n^2\right)^k, \quad k = 1, 2, \ldots ,$$

where M_k is a constant depending only on k.

Denoting by $s_n(t)$ the partial sums of the series 5.5(1), we have

$$(2) \qquad \int_0^1 s_n^{2k}(t)\,dt = \sum A_{\alpha_1\, \alpha_2 \ldots \alpha_r}\; c_{m_1}^{\alpha_1} \ldots c_{m_r}^{\alpha_r} \int_0^1 \varphi_{m_1}^{\alpha_1} \ldots \varphi_{m_r}^{\alpha_r}\, dt,$$

where $A_{\alpha_1, \alpha_2, \ldots, \alpha_r} = (\alpha_1 + \alpha_2 + \ldots + \alpha_r)!/\alpha_1!\, \alpha_2! \ldots \alpha_r!$, and the summation on the right is taken over the set of $m_1, m_2, \ldots, m_r, \alpha_1, \alpha_2, \ldots, \alpha_r$ defined by the relations.

$$0 \leqslant m_i \leqslant n,\; 0 \leqslant \alpha_i \leqslant 2k,\; i = 1, 2, \ldots,\; r, 1 \leqslant r \leqslant 2k,\; \alpha_1 + \alpha_2 + \ldots + \alpha_r = 2k.$$

Now is it easily verified that the integrals on the right vanish unless $\alpha_1, \alpha_2, \ldots, \alpha_r$ are all even, in which case they are equal to 1. Thus the right-hand side of (2) may be written $\sum A_{2\beta_1, \ldots 2\beta_r}\, c_{n_1}^{2\beta_1} \ldots c_{n_r}^{2\beta_r}$. Observing that

$$\sum A_{\beta_1, \beta_2, \ldots, \beta_r}\, c_{m_1}^{2\beta_1}\, c_{m_2}^{2\beta_2} \ldots c_{m_r}^{2\beta_r} = (c_0^2 + c_1^2 + \ldots + c_n^2)^k,$$

we obtain (2) with $f(t)$ replaced by $s_n(t)$, M_k being now the upper bound of the ratio $A_{2\beta_1, \ldots, 2\beta_r}/A_{\beta_1, \ldots, \beta_r}$. Since $s_n(t) \to f(t)$ for almost every t, an appeal to Fatou's lemma completes the proof.

It is easy to see that $M_k \leqslant (2k)!/2^k\, k! = (k+1) \ldots 2k/2^k \leqslant k^k$. This enables us to strengthen the theorem which we have just proved and to show that

(ii) *The function* $\exp \mu f^2(t)$ *is integrable for every* $\mu > 0$.

Let $C = c_0^2 + c_1^2 + \ldots$ Integrating the equation $\exp \mu f^2 = 1 + \mu f^2/1! + \mu^2 f^4/2! + \ldots$ over the interval $0 \leqslant t \leqslant 1$, and using the inequalities (1) with $M_k = k^k$, $k = 0, 1, \ldots$ we obtain that

$$(3) \qquad \int_0^1 \exp \mu f^2\, dt \leqslant \sum_{k=0}^{\infty} \frac{k^k}{k!} (\mu C)^k.$$

In virtue of Stirling's formula $k! \simeq 2\pi\, e^{-k}\, k^{k+1/2}$, the series on the right is certainly convergent if $e\mu C < 1$, that is if C is small enough. It follows that, for every $\mu > 0$, the function $\exp \mu (f - s_n)^2$ is integrable if only $n = n(\mu)$ is large enough. Since $f^2 \leqslant 2\,[(f - s_n)^2 + s_n^2]$, and $s_n(t)$ is bounded, the integrability of $\exp \mu f^2$ follows.

5.6. Applications of Rademacher's functions [1]). The theorems established in the preceding paragraph enable us to prove some results about the series

(1) $$\pm \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} \pm (a_n \cos nx + b_n \sin nx),$$

which we obtain from the standard series

(2) $$\tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

by changing the signs of terms of the latter in a quite arbitrary manner. Let $\frac{1}{2} a_0 = A_0(x)$, $a_n \cos nx + b_n \sin nx = A_n(x)$, $n = 1, 2, \ldots$ Neglecting the sequences $\{\pm 1\}$ containing only a finite number of $+1$ or of -1, we may present the series (1) in the form

(3) $$\sum_{n=0}^{\infty} A_n(x)\, \varphi_n(t),$$

where φ_n are Rademacher's functions and the parameter t, $t \neq p/2^q$, runs through the interval $(0, 1)$. If the values of t for which the series (3) possess a property P form a set of measure 1, we shall say that almost all the series (1) possess the property P.

(i) *If the series*

(4) $${}^1/_4\, a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

converges, then almost all the series (1) *converge almost everywhere in the interval* $0 \leqslant x \leqslant 2\pi$. (ii) *If the series* (4) *diverges, then, whatever method* T^* *of summation we consider, almost all the series* (1) *are almost everywhere non-summable* T^*.

Let $S_t(x)$ denote the series (3), and, if the series converges, let $S_t(x)$ also denote the sum. Let E be the set of points (x, t) in the rectangle $0 \leqslant x \leqslant 2\pi$, $0 \leqslant t \leqslant 1$, where the series converges. Assuming that the series (4) converges, we obtain from Theorem 5.5 (i) that the intersection of E with every line $x = x_0$, $0 \leqslant x_0 \leqslant 2\pi$, is of measure 1. Since the set E is measurable, its plane measure is 2π, and therefore the intersection of E with almost every line $t = t_0$, $0 \leqslant t_0 \leqslant 1$, is of measure 2π; this is just the first part of the theorem. The second part is proved by the same argument provided we can show that the divergence of (4) implies the divergence of

---

1) Paley and Zygmund [1].

$$(5) \qquad A_0^2(x) + A_1^2(x) + \dots + A_n^2(x) + \dots$$

for almost every x.

To establish the latter proposition suppose that the series (5) converges in a set of positive measure. Then there exists a set H, $|H| > 0$, and a constant M such that the sum of the series (5) does not exceed M for $x \in H$. Put $A_n(x) = \rho_n \cos(nx + x_n)$, $\rho_n \geqslant 0$. The series (5) may be integrated over H and we have

$$\sum_{n=1}^{\infty} \rho_n^2 \int_H \cos^2(nx + x_n)\, dx \leqslant M\,|H|.$$

The coefficients of ρ_n^2 in this inequality tend to $\frac{1}{2}|H|$ and so all of them exceed an $\varepsilon > 0$. Thence we conclude that the series $\rho_1^2 + \rho_2^2 + \dots$, i. e. the series (4), converges, contrary to our hypothesis.

The following proposition is an immediate corollary of (ii).

If the series (3) *diverges, almost all the series* (1) *are not Fourier series.*

The theorem of Riesz-Fischer asserts that, if (4) is finite, the series (2) is a Fourier series. Now we see that the Riesz-Fischer theorem is, in a way, the best possible: *no condition on the moduli of a sequence* $\{a_n, b_n\}$ *which permits* (4) *to diverge can possibly be a sufficient condition for* (2) *to be a Fourier series* [1]).

(iii) *If* (4) *is finite, then almost all the series* (1) *belong to* L^q *for every* $q > 0$. *More generally, for almost every* t *the function* $\exp \mu\, S_t^2(x)$ *is integrable over the interval* $0 \leqslant x \leqslant 2\pi$, *however large* μ *may be.*

Let C denote the sum of the series (4), and let μ be so small that the series in 5.51(3) converges. If $K = K(\mu, C)$ is the sum of the latter series, we have, as in 5.51(3),

$$\int_0^1 \exp \mu\, S_t^2(x)\, dt \leqslant K.$$

Integrating this inequality over the range $0 \leqslant x \leqslant 2\pi$ and interchanging the order of integration, we find that

$$(6) \qquad \int_0^{2\pi} dx \int_0^1 \exp \mu\, S_t^2(x)\, dt = \int_0^1 dt \int_0^{2\pi} \exp \mu\, S_t^2(x)\, dx \leqslant 2\pi K.$$

---

[1]) Littlewood [1], [2].

The interchanging of the order of integration is legitimate since the integrand is positive.

From (6) we conclude that the integral $\int_0^{2\pi} \exp \mu S_t^2(x)\, dx$ is finite for almost every t. This establishes the theorem for μ positive and sufficiently small. To establish the general result we argue as in the proof of Theorem 5.51(ii).

5.61. Let 5.6(4) be finite. In this case it is natural to ask whether the functions $S_t(x)$ are continuous functions of x for almost all t. But this is not so. In Chapter VI we shall prove that if a lacunary trigonometrical series is the Fourier series of a bounded function, the series of coefficients converges absolutely. Thus for no sequence of signs is the series

(1) $$\pm \sin 10\, x \pm \frac{1}{2} \sin 10^2 x + \ldots \pm \frac{1}{n} \sin 10^n x \pm \ldots$$

the Fourier series of a bounded function.

If the series

(2) $$\sum_{k=2}^{\infty} (a_k^2 + b_k^2) \log^{1+\varepsilon} k$$

converges for an $\varepsilon > 0$, *then almost all the series* 5.6(1) *are Fourier series of continuous functions.*

As the series (1) shows, the theorem is not true for $\varepsilon = 0$.

We require two lemmas.

(i) *Let* $\sigma_{n,t}(x)$ *denote the* $(C, 1)$ *means of the series* 5.6(3). *If the series* 5.6(4) *is finite, then, for almost every t, we have* $\sigma_{n,t}(x) = o(\sqrt{\log n})$, *uniformly in x.*

Let us put $\Phi(x) = \exp \mu x^2 - 1$, $\mu \geqslant 1$, $\varphi(x) = \Phi'(x) = 2\mu x \exp \mu x^2$. We will obtain an inequality for the function $\Psi(x)$ complementary to $\Phi(x)$ (§ 4.11). Let $x = \psi(y)$ be the function inverse to $y = \varphi(x)$. Since $\log \varphi(x) = \log 2\mu x + \mu x^2 \geqslant \mu x^2$ for $x \geqslant 1$, we see that $x = \psi(y) \leqslant \mu^{-1/2} \sqrt{\log y}$ whenever $x \geqslant 1$. Let y_0 be the root of the equation $\psi(y_0) = 1$. It follows that $\psi(y) \leqslant 1$ for $0 \leqslant y \leqslant y_0$, and $1 \leqslant \psi(y) \leqslant \mu^{-1/2} \sqrt{\log y}$ for $y \geqslant y_0$. Thence we deduce that $\Psi(y) \leqslant y$ for $y \leqslant y_0$, and $\Psi(y) \leqslant \mu^{-1/2} y \sqrt{\log y}$ for $y \geqslant y_0$, i. e. $\Psi(y) \leqslant y \chi(y)$, where $\chi(y) = \text{Max}\,(1, \mu^{-1/2} \sqrt{|\log y|})$.

Applying Young's inequality to the integral defining $\sigma_{n,t}(x)$, we see that

(3) $$\pi\, |\sigma_{n,t}(x)| \leqslant \int_0^{2\pi} \Phi\, |S_t(u)|\, du + \int_0^{2\pi} \Psi\, \{K_n(u - x)\}\, du,$$

where K_n denotes Fejér's kernel. Since $K_n \leqslant n$, the second integral on the right is less than

$$\left(\frac{\log n}{\mu}\right)^{1/2} \int_0^{2\pi} K_n(u - x)\, du = \pi \left(\frac{\log n}{\mu}\right)^{1/2},$$

provided that $\log n > \mu$. Taking t such that $\exp \mu S_t^2(u)$ is integrable for every value of μ, we see that the first integral on the right in (2) is finite, and so the left-hand side of (3) is certainly less than $2\pi\mu^{-1/2}\sqrt{\log n}$ if n is large enough. Since we may take μ as large as we please, the lemma follows.

(ii) *If the first arithmetic means for the series* 5.6(2) *are* $O(\log n)^{1/2}$, *the series*

$$(4) \qquad \sum_{n=2}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{(\log n)^{1/2+\varepsilon}}, \qquad \varepsilon > 0,$$

is uniformly summable $(C, 1)$.

Let us put $c_0 = c_1 = 0$, $c_\nu = (\log \nu)^{-1/2-\varepsilon}$ for $\nu \geqslant 2$, $h_\nu = h_\nu^{(n)} = (n+1-\nu)/(n+1)$, $C_\nu = c_\nu h_\nu$, and let $\sigma_n(x)$, $\tau_n(x)$ denote the first arithmetic means for the series 5.6(2) and (4) respectively. Applying Abel's transformation twice we obtain

$$(5) \qquad \tau_n(x) = \sum_{\nu=0}^{n-1} (\nu+1)\sigma_\nu(x) \Delta^2 C_\nu + (n+1)\sigma_n(x) \Delta C_n.$$

Since $\Delta C_n = C_n$, the last term on the right is $o(1)$ uniformly in x.

The reader will have no difficulty in proving the formula $\Delta^2 C_\nu = h_\nu \Delta^2 c_\nu + 2\Delta h_\nu \Delta c_{\nu+1} + \Delta^2 h_\nu c_{\nu+2}$, which is analogous to the formula for the second derivative of the product of two functions. In our case $\Delta^2 h_\nu = 0$ and so, by (5),

$$(6) \qquad \tau_n(x) = \sum_{\nu=0}^{n-1} h_\nu^{(n)} (\nu+1)\sigma_\nu(x) \Delta^2 c_\nu + \frac{2}{n+1} \sum_{\nu=0}^{n-1} (\nu+1)\sigma_\nu(x) \Delta c_{\nu+1} + o(1).$$

Given any function $\lambda(x)$, let us put $\alpha(u) = \alpha_x(u) = \lambda(x) - \lambda(x+u)$, $\beta(u) = \beta_x(u) = \lambda(x) - 2\lambda(x+u) + \lambda(x+2u)$. Since $\alpha(0) = \beta(0) = \beta'(0) = 0$ we obtain, by Taylor's formula, that $\alpha(u) = -\lambda'(x+\theta u)$, $\beta(u) = \frac{1}{2}\lambda''(x+\theta_1 u)$, where $0 < \theta < 1$, $0 < \theta_1 < 1$. Taking $\lambda(x) = (\log x)^{-1/2-\varepsilon}$, $u = 1$, we obtain that $\alpha_\nu(1) = \Delta c_\nu = O(\nu^{-1} \log^{-3/2-\varepsilon} \nu)$, $\beta_\nu(1) = \Delta^2 c_\nu = O(\nu^{-2} \log^{-3/2-\varepsilon} \nu)$. Thence we see that $(\nu+1)\sigma_\nu(x) \Delta c_{\nu+1} \to 0$, and, by (6),

$$(7) \qquad \tau_n(x) = \sum_{\nu=0}^{n-1} h_\nu^{(n)} (\nu+1)\sigma_n(x) \Delta^2 c_\nu + o(1) = \sum_{\nu=0}^{n} h_\nu^{(n)} (\nu+1)\sigma_\nu(x) \Delta^2 c_\nu + o(1).$$

Since the partial sums of the series with terms $(\nu+1)\sigma_\nu(x)\Delta^2 c_\nu$ are uniformly convergent, the same is true for the first Cesàro means, so that the last sum in (7) converges uniformly, and the lemma is established.

To complete the proof of the theorem let $a_n' = a_n(\log n)^{\frac{1+\varepsilon}{2}}$, $b_n' = b_n(\log n)^{\frac{1+\varepsilon}{2}}$. In virtue of (i), the first arithmetic means of almost all series with terms $\pm(a_n' \cos nx + b_n' \sin nx)$ are $o\sqrt{\log n}$, so that, by (ii), almost all series with terms $\pm(a_n \cos nx + b_n \sin nx)$ are uniformly summable $(C, 1)$, i. e. belong to the class C.

We add that this theorem can be generalized, viz. *if* (2) *is finite, almost all the series* 5.6(1) *converge uniformly over* $(0, 2\pi)$ [1].

---

[1]) Paley and Zygmund [1]; see also Salem [2].

5.7. Miscellaneous theorems and examples.

1. Let $\{a_n\}$ be a sequence tending to 0. A necessary and sufficient condition that $\{a_n\}$ should be quasi-convex is that it should be a difference of two convex sequences tending to 0.

If $\{a_n\}$ tends to 0 and is quasi-convex, then the sequences $\{a_n\}$ and $\{n\Delta a_n\}$ are of bounded variation.

2. If we put $f(x) = \sum n^{-\alpha} \cos nx$, $g(x) = \sum n^{-\alpha} \sin nx$, $0 < \alpha < 1$, then $f(x) \simeq x^{\alpha-1} \sin \frac{1}{2}\pi\alpha\, \Gamma(1-\alpha)$, $g(x) \simeq x^{\alpha-1} \cos \frac{1}{2}\pi\alpha\, \Gamma(1-\alpha)$ as $x \to +0$.

[This follows from the first formula in 3.11(1) and from the fact that $A_n^\beta \Gamma(\beta+1)/n^\beta = 1 + O(1/n)$ (§ 3.12)].

3. Let $g_k(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{\sin nh}{nh}\right)^k \cos nx$, $k = 1, 2, \dots$, $0 < kh \leqslant \pi$. The function $g_k(x)$ vanishes in the interval (kh, π) and is equal to a polynomial of order $k-1$ in each of the intervals $((k-2)h, kh)$, $((k-4)h, (k-2)h)$, ...

[Consider the function $f_k(x) = \sum_{m=-\infty}^{+\infty}{}' \frac{e^{imx}}{i^k m^k}$ of § 2.15 and the expression $g_k(x+kh) - \binom{k}{1} g_k(x+(k-2)h) + \dots \pm f_k(x-kh)$.

The result may also be obtained by repeated application of Theorem 2.11 to the function $f_1(x)$ (§ 1.8.2i)].

4. If $a_n \geqslant a_{n+1} \to 0$, the series $\sum a_n \cos nx$ is a Fourier-Riemann series. Szidon [1].

5. If $a_n \geqslant a_{n+1} \to 0$ and $\sum a_n \sin nx \in L$, then $\sum a_n \cos nx \in L$.

6. (i) If $\{a_n\}$, $a_n \to 0$, is convex, the functions $f(x) = \sum a_n \cos nx$ and $\bar{f}(x) = \sum a_n \sin nx$ have continuous derivatives in any interval $(\varepsilon, \pi-\varepsilon)$, $\varepsilon > 0$. (ii) If $\{a_n\}$ is only monotonic, this is not necessarily true, and the functions may be almost everywhere non-differentiable.

[(i) follows from the fact that the series differentiated term by term are uniformly summable $(C, 1)$ in $(\varepsilon, \pi-\varepsilon)$. To prove (ii) observe that the second series in 5.121(1) behaves like a lacunary series if $\lambda_{n+1}/\lambda_n > \lambda > 1$ and apply the following theorem].

7. Let the series 5.4(1) be a $\mathfrak{S}[f]$. If $f'(x)$ exists and is finite in a set E of positive measure, then $\sum n_k^2 (a_k^2 + b_k^2) < \infty$.

[This follows from Theorem 5.4 since the differentiated series is summable in E by a method T^*].

8. Let $\varphi_0(t)$, $\varphi_1(t)$, ... be Rademacher's functions and let $\sum c_n^2 < \infty$, $f(t) = \sum c_n \varphi_n(t)$, $0 \leqslant t \leqslant 1$. Then $m_\alpha \mathfrak{N}_2[c] \leqslant \mathfrak{M}_\alpha[f] \leqslant M_\alpha \mathfrak{N}_2[c]$, $\alpha > 0$, where the constants m_α and M_α depend only on α.

[The second inequality follows from Theorem 5.51 and from the fact that $\mathfrak{M}_\alpha[f; 0, 1] = \mathfrak{M}_\alpha[f]$ is a non-decreasing function of α. To prove the first inequality for $0 < \alpha < 2$ observe that $\mathfrak{M}_\alpha^\alpha$ is a multiplicatively convex function of α].

9. Let 5.4(1) be a $\mathfrak{S}[f]$ and let $n_{k+1}/n_k > \lambda > 1$. Then we have the inequalities $m_{\alpha,\lambda}\,\mathfrak{N}_2[\rho] \leqslant \mathfrak{M}_\alpha[f; 0, 2\pi] \leqslant M_{\alpha,\lambda}\,\mathfrak{N}_2[\rho]$, where $\rho_n^2 = a_n^2 + b_n^2$ and the constants m and M depend only on α and λ.

[It is sufficient to prove, for lacunary series, a theorem analogous to Theorem 5.51(i). The proof is similar if, for fixed α, λ is sufficiently large. In the general case we split up the series considered into a finite number of series for each of which the number λ is large].

10. If the series 5.4(1), with $n_{k+1}/n_k > \lambda > 3$, converges in an interval (a, b), then the series converges absolutely. Fatou [1].

[Let $a_k \cos n_k x + b_k \sin n_k x = \rho_k \cos(n_k x + x_k)$. It is easily seen geometrically that there is a point x^* in (a, b) such that $\cos(n_k x^* + x_k) > \varepsilon > 0$ for k sufficiently large. The theorem holds for $\lambda > 1$. See Zygmund [6]].

11. The points of convergence and those of divergence for the series $\sum(\sin 10^n x)/n$ are everywhere dense in the interval $(0, 2\pi)$.

12. Let $0 < \alpha < 1$ and $0 < \beta$. The series $\sum_{n=1}^{\infty} n^{-\beta} e^{in^\alpha} e^{inx}$ converges everywhere if $\alpha + \beta > 1$; the convergence is uniform if $\frac{1}{2}\alpha + \beta > 1$. Hardy [1]

13. If $1 \leqslant \frac{1}{2}\alpha + \beta \leqslant 2$, the sum of the previous series belongs to $\mathrm{Lip}(\frac{1}{2}\alpha + \beta - 1)$. Hardy [1], Zygmund [7].

[Apply van der Corput's lemmas and an argument similar to that of § 5.32].

14. The function $F(x) = -x + \lim_{m\to\infty} \int_0^x \prod_{p=1}^{m} (1 + \cos 4^p t)\,dt$ is a continuous function of bounded variation with Fourier coefficients $\neq o(1/n)$. F. Riesz [5].

[The product $p_m = (1 + \cos 4t) \dots (1 + \cos 4^m t)$ is a trigonometrical polynomial of order $\alpha_m = 4^m + 4^{m-1} + \dots + 4$. Since the lowest term of the polynomial $p_{m+1} - p_m = p_m \cos 4^{m+1} t$ is of order $\beta_{m+1} = 4^{m+1} - 4^m - \dots - 4 > \alpha_m$, p_m is a partial sum of p_{m+1}, i. e. $\{p_m\}$ is a subsequence of the sequence of partial sums of a trigonometrical series (*) $1 + a_1 \cos x + a_2 \cos 2x + \dots$ Let $P_m(x)$ be the integral of p_m over the interval $(0, x)$, and let γ_m be the number of non-vanishing terms in p_m; it is easy to see that $\gamma_{m+1} = 3\gamma_m - 1$, i. e. $\gamma_{m+1} - \gamma_m = 3(\gamma_m - \gamma_{m-1})$, $\gamma_{m+1} - \gamma_m = 3^m$. Since $p_{m+1} - p_m$ consists of 3^m terms each of which does not exceed 1 in absolute value, we have $|P_{m+1} - P_m| \leqslant 3^m/\beta_{m+1} = O(3^m/4^m)$ and so the function $P(x) = \lim P_m(x) = P_1 + (P_2 - P_1) + \dots$ is continuous. $P_m(x)$ is non-decreasing and so is its limit. It follows that the function $F(x) = -x + P(x)$ is continuous and of bounded variation. To obtain $\mathfrak{S}[F]$ we reject the linear term from the series (*) integrated term by term. Since $a_{4^m} = 1$, the coefficients of $\mathfrak{S}[F]$ are not $o(1/n)$].

CHAPTER VI.

The absolute convergence of trigonometrical series.

6.1. The Lusin-Denjoy theorem. The convergence of the series

(1)
$$\sum_{n=1}^{\infty}(|a_n|+|b_n|)$$

implies the absolute convergence of the series

(2)
$$\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx).$$

The series (2) may be absolutely convergent at an infinite set of points without (1) being convergent. A simple example is given by the series $\Sigma\sin n!\,x$, whose terms vanish from some place onwards for every x commensurable with π.

If the series (2) *converges absolutely in a set E of positive measure, the series* (1) *converges* [1]). Suppose, for simplicity, that $a_0=0$, and let $a_k\cos kx+b_k\sin kx=\rho_k\cos(kx+x_k)$, where $\rho_k\geqslant 0$, $\rho_k^2=a_k^2+b_k^2$. The function

(3)
$$\alpha(x)=\sum_{n=1}^{\infty}\rho_n|\cos(nx+x_n)|$$

is finite at every point of E. Hence there exists a set $\mathcal{E}\subset E$, $|\mathcal{E}|>0$, such that $\alpha(x)$ is bounded on $\mathcal{E}$, $\alpha(x)<M$ say. Since the partial sums $\alpha_n(x)$ of (3) are uniformly bounded on $\mathcal{E}$, the series may be integrated formally over $\mathcal{E}$:

(4)
$$\sum_{n=1}^{\infty}\rho_n\int_{\mathcal{E}}|\cos(nx+x_n)|\,dx\leqslant M|\mathcal{E}|.$$

---

[1]) Lusin [3], Denjoy [1].

To prove the convergence of $\rho_1+\rho_2+\dots$, which is equivalent to our theorem, it is sufficient to show that the integrals I_n on the left in (4) all exceed an $\varepsilon>0$. Let I'_n be the integral analogous to I_n, with $|\cos(nx+x_n)|$ replaced by $\cos^2(nx+x_n)$. Since $I_n>I'_n$, it is sufficient to prove that $I'_n>\varepsilon$. For this purpose we use the formula $2\cos^2(nx+x_n)=1+\cos 2nx\cdot\cos 2x_n-\sin 2nx\cdot\sin 2x_n$. Since the Fourier coefficients of the characteristic function of the set $\mathcal{E}$ tend to 0, we obtain that $I'_n\to\frac{1}{2}|\mathcal{E}|$, which completes the proof, all I'_n being positive.

The set E in the theorem which we have established is of positive measure. This property, while sufficient for the convergence of (1), is not necessary. The problem of necessary and sufficient conditions seems to be unsolved.

6.11. We shall supplement the previous theorem by a few results of the same character. Suppose that, for the series 6.1(2), we have $\rho_1+\rho_2+\dots=\infty$, and let E be the set of points where $\alpha(x)<\infty$. The complementary set H, where the upper limit of the sequence $\{\alpha_n(x)\}$ of continuous functions is equal to $+\infty$, is a product of a sequence of open sets; for if G_N denotes the open set of points where at least one of the functions $\alpha_n(x)$ exceeds N, we have $H=G_1G_2\dots$ It follows that E is the sum of a sequence of closed sets. None of these closed sets contains an interval; for otherwise, we should have $\rho_1+\rho_2+\dots<\infty$. It follows that all of them are non-dense, E is of the first category, and therefore, *if* 6.1(2) *converges absolutely in a set of the second category, even if it is of measure* 0, *the series* 6.1(1) *converges*[1]).

6.12. There exist trigonometrical series absolutely convergent in a perfect set but not everywhere (§ 6.6.1). On the other hand, as we shall prove, there exist perfect sets P of measure 0, which, as regards the absolute convergence of trigonometrical series, resemble sets of positive measure: every trigonometrical series absolutely convergent in P is absolutely convergent everywhere. In particular Cantor's well-known set has this property.

A point-set B will be called a *basis*, if every real x can be represented in the form $\alpha_1x_1+\alpha_2x_2+\dots+\alpha_lx_l$, where $\alpha_1,\alpha_2,\dots$ are integers, and $x_1,x_2,\dots$ belong to B. We may also write

---

[1]) Lusin [1].

$x = \varepsilon_1 x_1 + ... + \varepsilon_m x_m$, where $\varepsilon_j = \pm 1$ and the x_j are not necessarily different. We require the following lemma.

Let B be a basis, and let $B^* = B^*_u$ denote the set B translated by a number u. There exists a set S of the second category such that, for every $y \in S$, we have $y = \alpha_1 x_1^* + \alpha_2 x_2^* + ... + \alpha_n x_n^*$, with α_j integral and $x_j^* \in B^*$[1]. To prove this, we observe that for every x we have $x = \alpha_1(x_1^* - u) + \alpha_2(x_2^* - u) + ...$, i. e. $x + ku = \alpha_1 x_1^* + \alpha_2 x_2^* + ...$, where $k = k_x$ is an integer. Let $E_n, -\infty < n < +\infty$, denote the set of x for which $k_x = n$. For any x may exist several k_x; we choose one of them. At least one of these sets, say E_{n_0}, is not of the first category, and we may take for S the set E_{n_0} translated by $n_0 u$. We may say that B^* is a bais for S.

If B is a basis, every trigonometrical series absolutely convergent in B is absolutely convergent everywhere[2]. Suppose first that the trigonometrical series considered contains only sine terms. We prove by induction that $|\sin n(\varepsilon_1 x_1 + ... + \varepsilon_m x_m)| \leqslant |\sin nx_1| + |\sin nx_2| + ... + |\sin nx_m|$, if $\varepsilon_j = \pm 1$, and the result follows. In the general case let u be any point of B, and let $x = y + u$. We have $a_n \cos nx + b_n \sin nx = a_n(u) \cos ny + b_n(u) \sin ny$, where $a_n(u) = a_n \cos nu + b_n \sin nu$, $b_n(u) = b_n \cos nu - a_n \sin nu$.

The absolute convergence of the series at the point $y = 0$ implies $|a_1(u)| + |a_2(u)| + ... < \infty$, and therefore the series $b_1(u) \sin y + b_2(u) \sin 2y + ...$ converges absolutely in a set B^* obtained from B by translating it by $-u$. In virtue of the lemma, B^* is a basis for a set S of the second category. The argument which we applied to sine series shows that $b_1(u) \sin y + b_2(u) \sin 2y + ...$ is absolutely convergent in S, and consequently, by Theorem 6.11, everywhere. The same may be said of the series with terms $a_n(u) \cos ny + b_n(u) \sin ny = a_n \cos nx + b_n \sin nx$, and the theorem is established.

6.13. To give an example, we shall show that *the Cantor ternary set C constructed on* $(0, 1)$ (or on any other interval) *is a basis*. More precisely, we will show that the set of all possible sums $x + y$, with $x \in C$, $y \in C$, fills up the whole interval $(0, 1)$[3]. This could be deduced from the fact that the ternary development of any $x \in C$ can be written in the form not containing the digit 1,

---

[1] Thence it is not difficult to deduce that B^* is itself a basis (§ 6.6.2), but this is not necessary for our purposes.

[2] See Niemytzki [1], for the case of sine series.

[3] Steinhaus [4]. More general results will be found in Denjoy [2], Mirimanoff [1].

but a geometrical proof is more illuminating. Consider the set Γ of points (x,y) of the plane such that $x \in C$, $y \in C$. The set Γ may be obtained by the following procedure. Divide the square Q_0 with opposite corners at $(0,0)$ and $(1,1)$ into nine equal parts, and, removing the interior of the five squares forming a cross, consider the sum Q_1 of the remaining four corner squares. For any of these corner squares we repeat our procedure, and let Q_2 be the sum of the new corner squares, and so on. Plainly $\Gamma = Q_0 Q_1 Q_2 \ldots$ The projection of any Q_i on the diagonal joining the points $(0,0)$ and $(1,1)$ fills up this diagonal. In other words, any straight line L_h with the equation $x+y=h$, $0 \leqslant h \leqslant 1$, meets every Q_i at one point at least. Since the Q_i are closed and form a decreasing sequence, $\Gamma L_h \neq 0$ for $0 \leqslant h \leqslant 1$, and this is just what we wanted to prove.

6.2. Fatou's theorems. The problem of the absolute convergence for sine or cosine series has a very simple solution in the case when the moduli of the coefficients form a decreasing sequence

If the series $a_1 \cos x + a_2 \cos 2x + \ldots$, $|a_1| \geqslant |a_2| \geqslant \ldots$, *is absolutely convergent at a point* x_0, *then* $|a_1| + |a_2| + \ldots < \infty$. *The same is true for the series* $a_1 \sin x + a_2 \sin 2x + \ldots$, *provided that* $x_0 \not\equiv 0$ $(\mathrm{mod}\ \pi)$ [1]. To prove the first part of the theorem we may plainly suppose that $0 < x_0 < \pi$. From the hypothesis it follows that $|a_1|\cos^2 x_0 + |a_2|\cos^2 2x_0 + \ldots < \infty$. Since $2\cos^2 nx_0 = 1 + \cos ny_0$, where $y_0 = 2x_0$, and since the series $|a_1|\cos y_0 + |a_2|\cos 2y_0 + \ldots$ converges (§ 1.23), the result follows. The second part is obtained by a similar argument.

6.21. The set A of points where a trigonometrical series 6.1(2) converges absolutely, possesses curious properties. Let $\bar{A}$ denote the set of points of absolute convergence for the series conjugate to 6.1(2), and let B and $\bar{B}$ be the sets of points where the series 6.1(2) and its conjugate converge, not necessarily absolutely. It will be convenient to place all these sets on the circumference of the unit circle.

Every point of A is a point of symmetry for the sets $A, \bar{A}, B, \bar{B}$ [2].

The proof follows from the formulae

$$a_n(x+h) + a_n(x-h) = 2a_n(x)\cos nh, \quad b_n(x+h) - b_n(x-h) = -2a_n(x)\sin nh,$$

---

[1]) Fatou [2]. The proof of the text is due to Saks.
[2]) Fatou [2].

where the notation is that of § 6.12. From the first of them we deduce that, if $|a_1(x)|+|a_2(x)|+\ldots<\infty$, and if the series $a_1(x-h)+a_2(x-h)+\ldots$ converges, or converges absolutely, so does the series $a_1(x+h)+a_2(x+h)+\ldots$

The theorem remains true if we consider the points of summability, the arcs of uniform convergence, etc.

6.22. *If A is infinite, then B, and similarly $\overline{B}$, is either of measure 0 or 2π* [1]). If $x \in A$, $x+h \in A$, then all the points $x+h$, $x+2h$, $x+3h, \ldots$ belong to A. Since A is infinite, h may be arbitrarily small, and so A is everywhere dense. Suppose that B and its complement C are both of positive measure, and let x_1 and x_2 be points of density 1 for B and C respectively. There exists an $\varepsilon>0$ such that, if any interval I, $|I|\leqslant 2\varepsilon$, contains x_1, we have $|IB|>\frac{1}{2}|I|$, and if any interval I', $|I'|\leqslant 2\varepsilon$, contains x_2, then $|I'C|>\frac{1}{2}|I'|$. Let $I=(x_1-\varepsilon, x_1+\varepsilon)$, and take an x_0 belonging to A and distant by less than $\frac{1}{2}\varepsilon$ from the middle-point of the arc (x_1, x_2). The set B reflected in x_0 goes into itself, and I into an interval I', $|I'|=2\varepsilon$, containing x_2. Since the inequalities $|I'B|>\frac{1}{2}|I'|$, $|I'C|>\frac{1}{2}|I'|$ are incompatible, we have a contradiction.

6.3. The absolute convergence of Fourier series. We begin by the following theorem due to S. Bernstein.

If $f \in \mathrm{Lip}\,\alpha$, $\alpha>\frac{1}{2}$, then $\mathfrak{S}[f]$ converges absolutely. For $\alpha=\frac{1}{2}$ this is no longer true [2]).

Suppose that 6.1(2) is $\mathfrak{S}[f]$. Then

(1)
$$f(x+h)-f(x-h)\sim 2\sum_{n=1}^{\infty} b_n(x)\sin nh,$$

$$\frac{1}{\pi}\int_0^{2\pi}[f(x+h)-f(x-h)]^2\,dx=4\sum_{n=1}^{\infty}\rho_n^2\sin^2 nh,$$

where $\rho_n^2=a_n^2+b_n^2$. The left-hand side of the last formula is $\leqslant Ch^{2\alpha}$, where $C, C_1, \ldots$ denote constants. On setting $h=\pi/2N$ we obtain two inequalities

(2)
$$\sum_{n=1}^{\infty}\rho_n^2\sin^2\frac{\pi n}{2N}\leqslant C\,N^{-2\alpha},\qquad \sum_{n=1}^{N}\rho_n^2\sin^2\frac{\pi n}{2N}\leqslant C\,N^{-2\alpha}.$$

---

[1]) Lusin [1].

[2]) Bernstein [2], [3].

Let us now assume that $N = 2^\nu$, $\nu = 1, 2, \ldots$ Taking into account only the terms with indices n exceeding $\frac{1}{2} N$, we obtain from the last inequality

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n^2 \leqslant 2C\, 2^{-2\nu\alpha}. \tag{3}$$

Thence, by Schwarz's inequality,

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n \leqslant \left(\sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n^2\right)^{1/2} \left(\sum_{n=2^{\nu-1}+1}^{2^\nu} 1^2\right)^{1/2} < C_1\, 2^{\nu(1/2-\alpha)}, \tag{4}$$

and finally

$$\sum_{n=2}^{\infty} \rho_n = \sum_{\nu=1}^{\infty} \sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n \leqslant C_1 \sum_{\nu=1}^{\infty} 2^{\nu(1/2-\alpha)}. \tag{5}$$

The last series is convergent since $\alpha > \frac{1}{2}$. The proof ot the second part of the theorem we postpone to § 6.33.

6.31. *If $f(x)$ is of bounded variation and belongs to* Lip α *for any positive α, $\mathfrak{S}[f]$ converges absolutely* [1]. That the second condition imposed on f is not superfluous is seen from the example of the series

$$\sum_{2}^{\infty} \frac{\sin nx}{n \log n}, \tag{1}$$

which, being the Fourier series of a function of bounded variation, indeed of an absolutely continuous function (§ 5.12), diverges absolutely (§ 6.2).

Let $\omega(\delta)$ be the modulus of continuity of f, and V the total variation of f over $(0, 2\pi)$. We start from the inequality

$$\sum_{k=1}^{2N} \left[f\left(x + \frac{k\pi}{N}\right) - f\left(x + (k-1)\frac{\pi}{N}\right)\right]^2 \leqslant$$
$$\leqslant \omega\left(\frac{\pi}{N}\right) \sum_{k=1}^{2N} \left| f\left(x + k\frac{\pi}{N}\right) - f\left(x + (k-1)\frac{\pi}{N}\right)\right| \leqslant V \omega\left(\frac{\pi}{N}\right),$$

which we integrate over $(0, 2\pi)$. On account of the periodicity,

---

[1]) Zygmund [8].

replacing x by $x+\xi$ does not affect the value of the integral, and so all integrals formed from the left-hand side are equal. Hence we have, by turns, with $N=2^\nu$,

$$2N\int_0^{2\pi}\left|f\left(x+\frac{\pi}{2N}\right)-f\left(x-\frac{\pi}{2N}\right)\right|^2 dx \leqslant 2\pi V\omega\left(\frac{\pi}{N}\right)$$

$$\sum_{n=1}^{\infty}\rho_n^2\sin^2\frac{\pi n}{2N}\leqslant C_2 N^{-\alpha-1}, \qquad \sum_{n=1}^{N}\rho_n^2\sin^2\frac{\pi n}{2N}\leqslant C_2 N^{-\alpha-1},$$

$$\sum_{n=2^{\nu-1}+1}^{2^\nu}\rho_n^2\leqslant 2C_2\,2^{-\nu(\alpha+1)}, \qquad \sum_{n=2^{\nu-1}+1}^{2^\nu}\rho_n\leqslant C_3\,2^{-\nu\alpha/2},$$

$$\sum_{n=2}^{\infty}\rho_n\leqslant C_3\sum_{\nu=1}^{\infty}2^{-\nu\alpha/2}<\infty.$$

6.32. The problem of the absolute convergence of trigonometrical series may be generalized as follows. Given a series 6.1(2), we ask about the values of the exponent β which makes

$$(1) \qquad \sum_{n=1}^{\infty}(|a_n|^\beta+|b_n|^\beta)$$

convergent. Theorem 6.3 is special a case of the following theorem; it is, in fact, the most important case of it.

If $f\in\text{Lip}\,\alpha$, $0<\alpha\leqslant 1$, *the series* (1) *converges for every* $\beta>2/(2\alpha+1)$, *but not necessarily for* $\beta=2/(2\alpha+1)$ [1].

The proof of the first part resembles the proof of the first part of Theorem 6.3. Let $\gamma=2/(2\alpha+1)$. Since $0<\gamma<2$, we may also assume that $0<\beta<2$. Starting with 6.3(3), and applying Hölders inequality, we obtain

$$\sum_{2^{\nu-1}+1}^{2^\nu}\rho_n^\beta\leqslant\left(\sum_{2^{\nu-1}+1}^{2^\nu}\rho_n^2\right)^{\beta/2}\left(\sum_{2^{\nu-1}+1}^{2^\nu}1\right)^{1-\beta/2}\leqslant C_4\,2^{\nu(1-\beta/\gamma)}$$

Here $1-\beta/\gamma<0$, and an argument similar to 6.3(5) yields the convergence of $\rho_2^\beta+\rho_3^\beta+\ldots$ or, what is the same thing, of the series (1). This gives the first part of the theorem.

6.33. The second part of Theorem 6.32, and of Theorem 6.3, is a simple corollary of the results obtained in § 5.3. It was

[1]) Szász [2].

proved there that the real and imaginary components of the first of the series

$$(1)\qquad \sum_{n=2}^{\infty}\frac{e^{in\log n}}{n^{1/2+\alpha}}e^{inx},\qquad 0<\alpha<1,\qquad \sum_{n=2}^{\infty}\frac{e^{in\log n}}{(n\log n)^{3/2}}e^{inx},$$

belong to Lip α, and it is easy to see that, for these components, the series with terms $\rho_n^{2/(2\alpha+1)}$ diverge. The components of the second series in (1) belong to Lip 1 (§ 5.33), and the series with terms $\rho_n^{2/3}$ diverges.

6.34. *If f is of bounded variation and also $f\in$ Lip α, $0<\alpha\leqslant 1$, the series* 6.32(1) *converges for* $\beta>2/(2+\alpha)$ [1].

The proof, which is analogous to that of Theorems 6.31 and 6.32, may be left to the reader (see also § 6.6.7).

6.35. Let $F(x)$ be an absolutely continuous and periodic function whose derivative $F'(x)=f(x)$ belongs to L^2.

If a_n, b_n are the Fourier coefficients of f, those of F will be $-b_n/n$, a_n/n. From the inequalities

$$\frac{|a_n|}{n}\leqslant\frac{1}{2}\left(a_n^2+\frac{1}{n^2}\right),\quad \frac{|b_n|}{n}\leqslant\frac{1}{2}\left(b_n^2+\frac{1}{n^2}\right),$$

we see that $\mathfrak{S}[F]$ converges absolutely. More generally, *if F is absolutely continuous and $F'\in L^p$, $p>1$, then $\mathfrak{S}[F]$ converges absolutely* [2]. The proof remains essentially the same as in the case $p=2$, if, instead Bessel's inequality, we use a more general inequality, due to Young, which will be established in Chapter IX. It is howewer much simpler to deduce the theorem from Theorem 6.31, observing that, if $F'\in L^p$, $p>1$, then F satisfies a Lipschitz condition of positive order (§ 4.7.3).

The result which we have established is, in turn, contained in the following theorem

6.36. (i) *If $F(x)$ is absolutely continuous, $F'(x)=f(x)$, and $|f|\log^+|f|$ is integrable, then $\mathfrak{S}[F]$ converges absolutely* [3]. It will be convenient to postpone the proof of (i) to Chapter VII, where we shall obtain this theorem as a corollary of the following important result due to Hardy and Littlewood:

---

[1]) Waraszkiewicz [1]; Zygmund [7].
[2]) Tonelli [2].
[3]) Zygmund [4].

(ii) *If $\mathfrak{S}[F]$ and $\overline{\mathfrak{S}}[F]$ are both Fourier series of functions of bounded variation, $\mathfrak{S}[F]$ converges absolutely.*

Here we only observe that the integrability of $|f|(\log^+|f|)^{1-\varepsilon}$, $\varepsilon>0$, would not be sufficient for the truth of (i). For if we take for $\mathfrak{S}[F]$ the series 6.31(1), which converges absolutely only at the points $x\equiv 0 \pmod{\pi}$, we have $f(x)\sim 1/x\log^2 x$ as $x\to+0$, (§ 5.221), so that $|f|(\log^+|f|)^{1-\varepsilon}$ is integrable for every $\varepsilon>0$.

6.4. Szidon's theorem on lacunary series. The following theorem on the absolute convergence of Fourier series bears a different character.

If a lacunary trigonometrical series

$$(1)\qquad \sum_{k=1}^{\infty}(a_k\cos n_k x+b_k\sin n_k x),\qquad n_{k+1}/n_k>\lambda>1,$$

is the Fourier series of a bounded function $f(x)$, $|f|\leqslant M$, the series converges absolutely [1]).

Taking, instead of $f(x)$, the functions $f(x)\pm f(-x)$, we may restrict ourselves to purely cosine or purely sine series, e. g. to the former. The idea of the proof consists in considering the non-negative polynomials

$$(2)\qquad P_l(x)=\prod_{k=1}^{l}(1+\varepsilon_k\cos m_k x),$$

where $\varepsilon_k=\pm 1$ and the positive integers m_k satisfy a condition $m_{k+1}/m_k>\mu\geqslant 3$. Multiplying out the product P_l we see that it consists of the constant term 1, and of terms $A_\nu\cos\nu x$, where $\nu=\pm m_{k_1}\pm\pm m_{k_2}\pm\ldots\pm m_{k_j}>0$, $m_{k_1}<m_{k_2}<\ldots<m_{k_j}\leqslant m_{k_l}$. From the last equation we see that ν is contained between $m_{k_j}(1-\mu^{-1}-\mu^{-2}-\ldots)$ and $m_{k_j}(1+\mu+\mu^2+\ldots)$, i. e. between $m_{k_j}(\mu-2)/(\mu-1)$ and $m_{k_j}\mu/(\mu-1)$. Therefore, since $\mu\geqslant 3$, the numbers $\pm m_{k_1}\pm\ldots\pm m_{k_j}$ corresponding to various sequences $\{k_i\}$ are all different; and, if μ is large enough, $\mu\geqslant\mu_0(\varepsilon)$, the indices ν corresponding to $A_\nu\neq 0$ concentrate in the neighbourhoods $(m_k(1-\varepsilon),\ m_k(1+\varepsilon))$ of the numbers m_k, where $\varepsilon>0$ is arbitrary.

Returning to the series (1), take ε so small that the intervals $(n_k(1-\varepsilon),\ n_k(1+\varepsilon))$, $k=1,2,\ldots$, do not overlap, and an integer r such that $\lambda^r>\mu_0(\varepsilon)$. Put $m_k^{(s)}=n_{kr+s}$, $k=1,2,\ldots$, $0\leqslant s\leqslant r-1$,

[1]) Szidon [2]; for a generalization see Zygmund [6].

and let $P_l^{(s)}(x)$ denote the polynomial (2) formed with $\{m_k^{(s)}\}$, $1 \leqslant k \leqslant l$, and $\varepsilon_k = \operatorname{sign} a_{kr+s}$. Since $m_{k+1}^{(s)}/m_k^{(s)} > \lambda^r > \mu_0(\varepsilon)$, we obtain

$$(3) \qquad \sum_{k=1}^{l} |a_{kr+s}| = \frac{1}{\pi}\int_0^{2\pi} f(x)\, P_l^{(s)}(x)\, dx \leqslant \frac{M}{\pi}\int_0^{2\pi} P_l^{(s)}\, dx = 2M,$$

since the constant term of $P_l^{(s)}(x)$ is equal to 1. Making $l \to \infty$, we find that each ot the r partial series into which we have decomposed the series $|a_r| + |a_{r+1}| + |a_{r+2}| + \dots$ converges. This completes the proof.

If (1) is a pure sine series, we consider, instead of (2), analogous polynomials, with cosines replaced by sines.

6.5. Wiener's theorem. It is obvious that the absolute convergence of $\mathfrak{S}[f]$ at a point x_0 is not a local property but depends on the behaviour of $f(x)$ in the whole interval $(0, 2\pi)$. However, *if to every point x_0 corresponds a neighbourhood I_{x_0} of x_0 and a function $g(x) = g_{x_0}(x)$ such that* (i) *$\mathfrak{S}[g]$ converges absolutely, and* (ii) *$g(x) = f(x)$ in I_{x_0}, then $\mathfrak{S}[f]$ converges absolutely*[1].

By the Heine-Borel theorem we can find a finite number of points $x_1, x_2, \dots x_m$ such that the intervals $I_{x_1}, I_{x_2}, \dots I_{x_m}$ overlap and cover the whole interval $0 \leqslant x \leqslant 2\pi$. Let $I_{x_k} = (u_k, v_k)$. Without loss of generality we may suppose that $u_k < v_{k-1} < u_{k+1} < v_k$, $k = 1, 2, \dots, m$, where $(u_{m+1}, v_{m+1}) = (u_1, v_1)$. Let $\lambda_k(x)$ be the periodic and continuous function equal to 1 in (v_{k-1}, u_{k+1}), vanishing outside (u_k, v_k) and linear in the intervals (u_k, v_{k-1}) and (u_{k+1}, v_k). It will be readily seen that $\lambda_1(x) + \lambda_2(x) + \dots + \lambda_m(x) = 1$. Since λ_k has a derivative of bounded variation, the Fourier coefficients of λ_k are $O(n^{-2})$, so that $\mathfrak{S}[\lambda_k]$ converges absolutely.

Since $\mathfrak{S}[f\lambda_k] = \mathfrak{S}[g_{x_k}\lambda_k] = \mathfrak{S}[g_{x_k}]\,\mathfrak{S}[\lambda_k]$, we obtain that $\mathfrak{S}[f\lambda_k]$ converges absolutely (§ 4.431). To prove the theorem it is sufficient to observe that $\mathfrak{S}[f] = \mathfrak{S}[f\cdot(\lambda_1 + \dots + \lambda_m)] = \mathfrak{S}[f\lambda_1] + \dots + \mathfrak{S}[f\lambda_m]$.

6.51[2]. *Let the Fourier series of a function $f(t)$ be absolutely convergent, and let the values of $f(t)$ belong to an interval (α, β). If $\varphi(z)$ is a function of a complex variable, regular at every point of the interval (α, β), the Fourier series of $\varphi\{f(t)\}$ converges absolutely.*

---

[1]) Wiener [1].

[2]) Lévy [1], Wiener [1].

Let $f(t)=\sum_{n=-\infty}^{+\infty} c_n e^{int}$, $f^k(t)=\sum_{n=-\infty}^{+\infty} c_n^{(k)} e^{int}$, $k=0, 1, 2, \ldots$ Since $\mathfrak{S}[f^k]$ is obtained from $\mathfrak{S}[f]$ by formal multiplication, it is easy to see that if $\ldots+|c_{-1}|+|c_0|+|c_1|+\ldots=M$, then $\ldots+|c_{-1}^{(k)}|+|c_0^{(k)}|+$ $+|c_1^{(k)}|+\ldots \leqslant M^k$. Suppose that the series $\varphi(z)=\alpha_0+\alpha_1 z+\alpha_2 z^2+\ldots$ converges for $|z|<r$. In the case $M<r$ the theorem is fairly simple; for the series $\alpha_0+\alpha_1 f(t)+\alpha_2 f^2(t)+\ldots$ converges uniformly, and, if γ_n are the complex Fourier coefficients of $\varphi\{f\}$, then

$$\gamma_n=\sum_{k=0}^{\infty}\alpha_k c_n^{(k)}, \quad \sum_{n=-\infty}^{+\infty}|\gamma_n| \leqslant \sum_{n=-\infty}^{+\infty}\sum_{k=0}^{\infty}|\alpha_k c_n^{(k)}|=\sum_{k=0}^{\infty}|\alpha_k|\sum_{n=-\infty}^{+\infty}|c_n^{(k)}|,$$

where the sum of the last series is $\leqslant|\alpha_0|+|\alpha_1|M+|\alpha_2|M^2+\ldots<\infty$.

Let t_0 be an arbitrary point of the interval $0\leqslant t\leqslant 2\pi$. To prove the theorem in the general case it is sufficient to show that there is a function $g(t)$ such that $\mathfrak{S}[\varphi\{g\}]$ converges absolutely and that $g(t)=f(t)$ in an interval $(t_0-h,\ t_0+h)$. Suppose, for simplicity, that $t_0=0$ and let $f(0)=u$. Without real loss of generality we may suppose that $u=0$, for otherwise we have $\varphi\{f(t)\}=\varphi\{f(t)-u+u\}=\varphi_1\{f_1(t)\}$, where $f_1(t)=f(t)-u$, $\varphi_1(z)=$ $=\varphi(z+u)$, and we may consider the functions f_1, φ_1 instead of f, φ.

Let $\varphi(z)=\alpha_0+\alpha_1 z+\ldots$ be convergent for $|z|<r$. In virtue of the special case already dealt with, it is sufficient to construct a function $g(t)$ with Fourier coefficients c'_n, such that $g(t)=f(t)$ in $(-h, h)$ and that $\ldots+|c'_{-1}|+|c'_0|+|c'_1|+\ldots=M'<r$; for then $\mathfrak{S}[\varphi\{g\}]$ will be absolutely convergent.

Let $\lambda(t)=\lambda_\rho(t)$ be a continuous periodic function such that (i) $\lambda(t)=1$ for $0\leqslant t\leqslant\rho$, (ii) $\lambda(t)=0$ for $2\rho\leqslant t\leqslant\pi$, (iii) $\lambda(t)$ is linear in the interval $(\rho, 2\rho)$, (iv) $\lambda(t)$ is even. If $l_n=l_n^\rho$ are the complex Fourier coefficients of $\lambda(t)$, then $l_0=3\rho/2\pi$, $l_n=(2\sin{}^1\!/_2\,\rho n\sin{}^3\!/_2\,\rho n)/\pi\rho n^2$, $n\neq 0$. We shall require the following relations

$$(1)\quad \sum_{n=-\infty}^{+\infty}|l_n^\rho|\leqslant A, \qquad (2)\quad \sum_{n=-\infty}^{+\infty}|l_n^\rho-l_{n-1}^\rho|\to 0 \quad \text{as } \rho\to 0,$$

where $A, B, \ldots$ denote constants independent of ρ. To prove (1) we observe that from the inequalities $|\sin u|\leqslant 1$, $|\sin u|\leqslant|u|$, we obtain $|l_n|\leqslant 2/\pi\rho n^2$, $|l_n|\leqslant 3\rho/2\pi$, and so, if $N=[1/\rho]+1$, the sum in (1) is less than

$$\frac{3\rho}{2\pi}+2\sum_{n=1}^{N}\frac{3\rho}{2\pi}+2\sum_{n=N+1}^{\infty}\frac{2}{\pi\rho n^2}<1+N\rho+4/\pi\rho N<A.$$

Now $l_n^\rho - l_{n-1}^\rho$ is the complex Fourier coefficient of the function $\lambda_\rho(t)(1-e^{it})$. Considering the real and imaginary parts of the derivative $\lambda'_\rho(t)(1-e^{it}) - ie^{it}\lambda_\rho(t)$ of this function, we easily find that the total variation of this derivative over $(-\pi, \pi)$ is uniformly bounded, and so, in virtue of the results obtained in § 2.213, we have $|l_n^\rho - l_{n-1}^\rho| \leqslant B/n^2$. If ν is a positive integer, the series in (2) is equal to

$$\sum_{n=-\nu}^{\nu} + \left(\sum_{n=-\infty}^{-\nu-1} + \sum_{n=\nu+1}^{\infty}\right) \leqslant \sum_{n=-\nu}^{\nu} |l_n^\rho - l_{n-1}^\rho| + 2B\sum_{n=\nu+1}^{\infty}\frac{1}{n^2} = P + Q.$$

Taking ν large enough we have $Q < \frac{1}{2}\epsilon$. If $\rho \to 0$, then $\mathfrak{M}[\lambda_\rho] \to 0$, and so $l_n^\rho \to 0$ for every n. Hence, for fixed ν, $P < \frac{1}{2}\epsilon$, $P + Q < \epsilon$ if ρ is small enough, and this proves (2).

Let $q > 0$ be an integer which we shall define in a moment, and let $c_p = u_p + v_p$, where $u_p = c_p$, $v_p = 0$ for $|p| \leqslant q$, and $u_p = 0$, $v_p = c_p$ for $|p| > q$. Since $f(0) = \Sigma c_p = 0$, $\Sigma |c_p| < \infty$ we have

$$\left|\sum_{p=-\infty}^{+\infty} u_p\right| < r/3A, \quad \sum_{p=-\infty}^{+\infty} |v_p| < r/3A$$

if q is large enough. Denoting by d_n^ρ the Fourier coefficients of the function $f(t)\lambda_\rho(t)$, we have

$$d_n^\rho = \sum_{p=-\infty}^{+\infty} c_p\, l_{n-p}^\rho,$$

$$\sum_{n=-\infty}^{+\infty} |d_n^\rho| \leqslant \sum_{n=-\infty}^{+\infty}\left|\sum_{p=-\infty}^{+\infty} u_p\, l_{n-p}\right| + \sum_{n=-\infty}^{+\infty}\left|\sum_{p=-\infty}^{+\infty} v_p\, l_{n-p}\right| = S + T,$$

$$T \leqslant \sum_{n=-\infty}^{+\infty}\sum_{p=-\infty}^{+\infty} |v_p|\,|l_{n-p}^\rho| = \sum_{p=-\infty}^{+\infty}|v_p|\sum_{n=-\infty}^{+\infty}|l_n^\rho| < \frac{r}{3A}\cdot A = {}^1/_3\, r,$$

$$S = \sum_{n=-\infty}^{+\infty}\left|\sum_{p=-q}^{q} c_p(l_{n-p} - l_n + l_n)\right| \leqslant \sum_{n=-\infty}^{+\infty}\left|\sum_{p=-q}^{q} c_p(l_{n-p} - l_n)\right| +$$

$$+ \sum_{n=-\infty}^{+\infty}|l_n|\left|\sum_{p=-q}^{q} c_p\right| = S_1 + S_2.$$

It it plain that $S_2 < {}^1/_3\, r$. Since $|l_{n-p} - l_n| \leqslant |l_{n-p} - l_{n-p+1}| + \ldots + |l_{n-1} - l_n|$ for $p > 0$, $|l_{n-p} - l_n| \leqslant |l_{n-p} - l_{n-p-1}| + \ldots + |l_{n+1} - l_n|$ for $p < 0$, S_1 is less than a multiple of the series (2) and so tends to 0 with ρ. If $\rho = \rho_0$ is small enough, then $S_1 < {}^1/_3\, r$,

$S + T \leqslant S_1 + S_2 + T < {}^1/_3\, r + {}^1/_3\, r + {}^1/_3\, r = r$. Hence, putting $g(t) = f(t)\,\lambda_{\rho_0}(t)$, $c'_n = d_n^{\rho_0}$, $h = \rho_0$, we shall have $f(t) = g(t)$ in $(-h, h)$, $\Sigma\,|c'_n| < r$, and this completes the proof.

As a corollary we obtain that, *if* $\mathfrak{S}[f]$ *converges absolutely and* $f(x) \neq 0$, *then* $\mathfrak{S}[1/f]$ *converges absolutely.*

6.6. Miscellaneous theorems and examples.

1. The set of points where the series $\Sigma\, n^{-1} \sin n!\, x$ converges absolutely contains a perfect subset.

[Consider the graphs of the curves $y = \sin n!\, x$].

2. (i) Every measurable set of positive measure is a basis; (ii) every set of the second category is a basis.

[Let E be an arbitrary set of positive measure, and $x \in E$, $y \in E$. To prove (i) it is sufficient to show that the set of the differences $x - y$ contains an interval. To show this let E_h denote the set E translated by h. Considering the neighbourhood of a point of density 1 for the set E, it is easy to show that $E\,E_h \neq 0$ if h is sufficiently small. This theorem is due to Steinhaus [5]. The proof of (ii) is similar].

3. A necessary and sufficient condition that the Fourier series of a function $h(x)$ should converge absolutely is that there should exist two functions f and g of the class L^2 such that $h(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x+t)\,g(t)\,dt$. M. Riesz; see Hardy and Littlewood [8].

[That the condition is sufficient follows from § 2.11. Let c_n be the complex Fourier coefficients of h; to prove that the condition is necessary consider the functions with Fourier coefficients $|c_n|^{1/2}$ and $|c_n|^{1/2}\,\overline{\text{sign}\, c_n}$].

4. The conditions of Theorems 6.3—6.32 are unnecessarily stringent. Thus Theorems 6.3 and 6.32 remain true, and the proofs unchanged, if we assume that $f \in \text{Lip}\,(\alpha, 2)$. In Theorem 6.31 we may assume that the function f is of bounded variation and belongs to Lip $(\alpha, 1)$.

5. Let $0 < \alpha \leqslant 1$, $1 \leqslant p \leqslant 2$. If a_n, b_n are the Fourier coefficients of an $f \in \text{Lip}\,(\alpha, p)$, then $\Sigma\,(|a_n|^\beta + |b_n|^\beta) < \infty$ for every $\beta < p/\{p(1+\alpha) - 1\}$. Szász [3].

[The proof is similar to that of Theorem 6.32 if, instead of Parseval's relation, we use the inequality of Hausdorff-Young which will be established in Chapter IX].

6. (i) If $f \in \text{Lip}\,\alpha$, $0 < \alpha \leqslant 1$, then $\Sigma\, n^{\beta - 1/2}(|a_n| + |b_n|) < \infty$ for every $\beta < \alpha$. Hardy [4]. (ii) If f is, in addition, of bounded variation then $\Sigma\, n^{\beta/2}(|a_n| + |b_n|) < \infty$. (iii) If $f \in \text{Lip}\,(\alpha, p)$, $0 < \alpha \leqslant 1$, $1 \leqslant p \leqslant 2$, then $\Sigma\, n^\gamma (|a_n| + |b_n|) < \infty$ for every $\gamma < \alpha - 1/p$.

[To prove the first part of the theorem consider the inequality 6.3(4)].

7. Let $f(x)=\sum_{n=2}^{\infty}\frac{e^{in^{\alpha}}}{n^{\frac{1}{2}\alpha+1}}\frac{e^{inx}}{(\log n)^{\beta}}$, where $0<\alpha<1$, and $\beta-1$ is positive and sufficiently small. Then the real and imaginary parts of f are of bounded variation, belong to Lip α, and yet $\sum\rho_n^k=\infty$ for $k=2/(2+\alpha)$. It follows that Theorem 6.34 cannot be improved. For the proof see Zygmund [7]. See also § 5.7.13.

8. Let a_k, b_k be the Fourier coefficients of a function $f(x)$ and let $t_n=t_n(f)=\frac{1}{2}\rho_1+\rho_2+\ldots+\rho_n$, where $\rho_k\geqslant 0$, $\rho_k^2=a_k^2+b_k^2$.

(i) If $|f(x)|\leqslant 1$, then $t_n\leqslant(2n+1)^{1/2}$. (ii) For every n there is a function $f(x)=f_n(x)$ such that $t_n\geqslant An^{1/2}$, where A is a positive absolute constant.

See Bernstein [3], where a little more is proved, viz. that for f we may take a trigonometrical polynomial of order n.

[(i) follows from the inequalities of Bessel and Schwarz. To obtain (ii) let $g_t(x)=g_{t,n}(x)=\varphi_1(t)\cos x+\ldots+\varphi_n(t)\cos nx$, where $\varphi_1,\varphi_2,\ldots$ are Rademacher's functions. Then

$$\int_0^1 dt\int_0^{2\pi}|g_t(x)|\,dx=\int_0^{2\pi}dx\int_0^1|g_t(x)|\,dt\geqslant m_1\int_0^{2\pi}(\cos^2 x+\ldots+\cos^2 nx)^{1/2}\,dx=$$

$$=\tfrac{1}{2}m_1\int_0^{2\pi}\{\cos^2 x+\ldots)^{1/2}+(\sin^2 x+\ldots)^{1/2}\}\,dx\geqslant\tfrac{1}{2}m_1\int_0^{2\pi}\{(\cos^2 x+\sin^2 x)+\ldots\}^{1/2}dx=\pi m_1 n^{1/2}$$

(§§ 5.7.8, 4.13(3)). Let t_0 be a value of t such that the integral of $|g_{t_0}(x)|$ over $(0,2\pi)$ exceeds $\pi m_1 n^{1/2}$, and let a_k, b_k be the Fourier coefficients of the function $f(x)=\operatorname{sign} g_{t_0}(x)$. Then

$$\sum_{k=1}^{n}(|a_k|+|b_k|)\geqslant\Big|\sum_{k=1}^{n}\varphi_k(t_0)(a_k+b_k)\Big|=\Big|\frac{1}{\pi}\int_0^{2\pi}f(x)\,g_{t_0}(x)\,dx\Big|=$$

$$=\frac{1}{\pi}\int_0^{2\pi}|g_{t_0}(x)|\,dx\geqslant m_1 n^{1/2}.$$

The idea of the proof is taken from Paley [2], where it is applied to another problem. The result may be used to prove the second part of Theorem 6.3].

CHAPTER VII.

Conjugate series and complex methods in the theory of Fourier series.

7.1. Summability of conjugate series [1]). In Chapter III we proved some results on the summability (C, r) of Fourier series. As regards the conjugate series our results were less complete. The obstacle was that we knew nothing about the existence of the integral

$$(1)\quad -\frac{1}{\pi}\int_0^\pi \frac{f(x+t)-f(x-t)}{2\operatorname{tg}\frac{1}{2}t}\,dt = \lim_{h\to 0}\left(-\frac{1}{\pi}\int_h^\pi \frac{f(x+t)-f(x-t)}{2\operatorname{tg}\frac{1}{2}t}\,dt\right).$$

We proved that, almost everywhere, the existence of (1) was equivalent to the summability A of $\tilde{\mathfrak{S}}[f]$. We now intend to prove the latter fact using complex methods, independently of the behaviour of (1). This will just enable us to prove the existence of (1) for almost every x. The proof will be based on the following lemma.

Let $G(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \ldots$, $z = re^{ix}$, *be a function which is regular, bounded, and non-vanishing in the circle* $|z| < 1$. *The function* $l(x) = \lim_{r\to 1} G(re^{ix})$ *may vanish only in a set of measure* 0.

Suppose that $|G(z)| < 1$. That $l(x)$ exists for almost every x follows from the fact that the real and imaginary parts of $\alpha_0 + \alpha_1 e^{ix} + \ldots$ are Fourier series of bounded functions (§ 4.36). Let us take any branch of the function $\log G(z) = \log|G(re^{ix})| + i \arg G(re^{ix})$. Since $G(z) \neq 0$ for $|z| < 1$, $\log G(z)$ is regular

---

[1]) Privaloff [2], Plessner [2], Hardy and Littlewood [4], Zygmund [2].

and $\log|G(re^{ix})| \leqslant 0$. It follows that the harmonic function $\log|G(re^{ix})|$ is a Poisson-Stieltjes integral (§ 4.36), and so tends, for almost every x, to a finite limit as $r \to 1$. This shows that $l(x) \neq 0$ for almost every x, and the lemma is established.

For any integrable $f(x)$, $\bar{\mathfrak{S}}[f]$ *is summable A almost everywhere.* It is sufficient to suppose that $f > 0$. Let $f(r, x)$ be the Poisson integral for $f(x)$, and $\bar{f}(r, x)$ the conjugate harmonic function. The values of the function $F(z) = f(r, x) + i\bar{f}(r, x)$, $z = re^{ix}$, belong to the right half-plane, so that the function $G(z) = 1/(F(z)+1)$ is regular and less than 1 in absolute value for $|z| < 1$. Hence, by the lemma, $\lim\limits_{r \to 1} G(re^{ix})$ exists and is different from 0 for almost every x. Thence we deduce that, for almost every x, $\lim F(re^{ix})$, and therefore $\lim \bar{f}(r, x)$, exists and is finite. As corollaries we obtain the following propositions.

(i) *For any integrable f the integral* (1) *exists almost everywhere.*

(ii) $\bar{\mathfrak{S}}[f]$ *is summable* (C, r), $r > 0$, *at almost every point, to the value* (1) (§ 3.32).

The integral (1) will be denoted troughout by $\bar{f}(x)$. The function $\bar{f}(x)$ is called the *conjugate* function of $f(x)$. Considering the points where $\mathfrak{S}[f]$ and $\bar{\mathfrak{S}}[f]$ are both summable $(C, 1)$, we obtain the following proposition (§ 3.14):

(iii) *Given any integrable $f(x)$, the conjugate harmonic function $\bar{f}(r, x)$, tends, for almost every x_0, to the value $\bar{f}(x_0)$ as the point (r, x) approaches $(1, x_0)$ along any path not touching the circle.*

7.11. *If $F(x)$, $0 \leqslant x \leqslant 2\pi$, is a function of bounded variation, $\bar{\mathfrak{S}}[dF]$ is, at almost every point, summable (C, r), $r > 0$, to the value*

$$(1) \quad -\frac{1}{\pi}\int_0^{\pi} \frac{F(x+t)+F(x-t)-2F(x)}{4\sin^2 \frac{1}{2}t}\,dt = \lim_{h \to 0}\left\{-\frac{1}{\pi}\int_h^{\pi} \frac{F(x+t)+F(x-t)-2F(x)}{4\sin^2 \frac{1}{2}t}\,dt\right\}.$$

The proof runs on the same lines as in the case when F is absolutely continuous. Supposing, as we may, that $F(x)$ is non-decreasing, let $f(r, x) > 0$ be the Poisson-Stjeltjes integral for dF, and $\bar{f}(r, x)$ the conjugate harmonic function. Since $\bar{f}(r, x) > 0$, we prove, as before, that $\lim\limits_{r \to 1} \bar{f}(r, x)$ exists and is finite for almost every x. Combining the arguments of §§ 3.45, 3.8, it can easily be shown (the details of the proof we leave to the reader) that, at any point where $F'(x)$ exists and is finite, $\bar{\mathfrak{S}}[dF]$ is summable A if and only if the integral (1) exists. An appeal to the second part of Theorem 3.8 completes the proof.

If F' is absolutely continuous and $F' = f$, an integration by parts shows the integrals (1) and 7.1(1) to be equal.

7.2. Conjugate series and Fourier series. We shall now be concerned with the very important problem of conditions under which the conjugate series is itself a Fourier series. A special result was established in § 4.22, but the method used there cannot be extended to more general cases. The following important result is due to M. Riesz.

7.21. *If* $f \in L^p$, $p > 1$, *then* $\bar{f} \in L^p$ *and there exists a constant* A_p *depending only on* p *and such that* $\mathfrak{M}_p[\bar{f}; 0, 2\pi] \leqslant A_p \mathfrak{M}_p[f; 0, 2\pi]$. *Moreover*, $\bar{\mathfrak{S}}[f] = \mathfrak{S}[\bar{f}]$ [1].

In virtue of Theorem 4.36 (iii), and of Fatou's lemma, the theorem which we have to prove is a corollary of, and is in reality equivalent to, the following proposition.

Let $F(z) = u(z) + iv(z)$, $v(0) = 0$, *be an arbitrary function regular inside the unit circle. Then*

$$(1) \qquad \mathfrak{M}_p[v(re^{ix})] \leqslant A_p \mathfrak{M}_p[u(re^{ix})], \quad 0 \leqslant r < 1, \quad p > 1.$$

It is sufficient to prove the truth of (1) in the case when $\Re f(z) = u(z) > 0$ for $|z| < 1$. In fact, having fixed r, let $\varphi_1(x) = \mathrm{Max}\{u(r, x), 0\}$, $\varphi_2(x) = \mathrm{Min}\{u(r, x), 0\}$, so that $u(re^{ix}) = \varphi_1(x) + \varphi_2(x) = \varphi(x)$, say. The functions φ_1, φ_2 are continuous and possess first derivatives which are continuous, except at a finite number of points where they have simple discontinuities. It follows that the conjugate functions $\bar{\varphi}(x)$, $\bar{\varphi}_1(x)$, $\bar{\varphi}_2(x)$ are also continuous. Let $\varphi(\rho, x)$, $\varphi_j(\rho, x)$, $\bar{\varphi}_j(\rho, x)$, $j = 1, 2$, denote the corresponding harmonic functions. Since $\varphi_j(\rho, x) > 0$, we have, assuming the truth of (1) for $u > 0$, that $\mathfrak{M}_p[\bar{\varphi}_j(\rho, x))] \leqslant A_p \mathfrak{M}_p[\varphi_j(\rho, x)]$, and, making $\rho \to 1$, $\mathfrak{M}_p[\bar{\varphi}_j(x)] \leqslant A_p \mathfrak{M}_p[\varphi_j(x)] \leqslant A_p \mathfrak{M}_p[\varphi(x)]$. By Minkowski's inequality we obtain: $\mathfrak{M}_p[\bar{\varphi}(x)] \leqslant \mathfrak{M}_p[\bar{\varphi}_1(x)] + \mathfrak{M}_p[\bar{\varphi}_2(x)] \leqslant 2A_p \mathfrak{M}_p[\varphi(x)]$. This is just (1) with the constant twice as large, which is, of course, immaterial.

Passing to the proof of the theorem, let us consider the branch of the function $F^p(z)$ which is positive at the origin. Writing u, v instead of $u(re^{ix})$, $v(re^{ix})$, we have, by Cauchy's theorem,

[1] M. Riesz [4].

$$(2)\qquad \frac{1}{2\pi i}\int_{|z|=r}\frac{F^p(z)}{z}\,dz=\frac{1}{2\pi}\int_0^{2\pi}(u+iv)^p\,dx=u^p(0)=\left\{\frac{1}{2\pi}\int_0^{2\pi}u\,dx\right\}^p.$$

The difference $(u+iv)^p-(iv)^p$ is equal to the integral of the function pz^{p-1}, which is regular in the right half-plane, taken along the straight line between iv and $u+iv$, and so its modulus does not exceed the length u of the path of integration, multiplied by the maximal modulus of the function integrated, viz. $p\,(u^2+v^2)^{1/2(p-1)}\leqslant p2^{1/2(p-1)}\,(u^{p-1}+v^{p-1})$. Using this and the fact that the last term in (2) is equal to $\mathfrak{A}^p[u]\leqslant\mathfrak{M}_p^p[u]$ (§ 4.15), we obtain from (2) the inequality

$$(3)\qquad \left|\frac{1}{2\pi}\int_0^{2\pi}(iv)^p\,dx\right|\leqslant\frac{p2^{1/2(p-1)}}{2\pi}\int_0^{2\pi}(u^p+u\,|v|^{p-1})\,dx+\frac{1}{2\pi}\int_0^{2\pi}u^p\,dx.$$

Now $(iv)^p=|v|^p\exp(\pm\frac{1}{2}\pi ip)$, where the sign in the exponent is that of v; it follows that $\Re\,(iv)^p=|v|^p\cos\frac{1}{2}p\pi$. Let I denote the integral on the left of (3). Then the inequality will hold à fortiori if we replace $|I|$ by $|\Re\,I|$; and so, applying Hölder's inequality to the product $u|v|^{p-1}$, we obtain the inequality $|\cos\frac{1}{2}p\pi|\,\mathfrak{M}_p[v]\leqslant p\,2^{1/2(p-1)}\,\{\mathfrak{M}_p^p[u]+\mathfrak{M}_p[u]\,\mathfrak{M}_p^{p-1}[v]\}+\mathfrak{M}_p^p[u]$. Denoting the ratio $\mathfrak{M}_p[v]/\mathfrak{M}_p[u]$ by X, we see that

$$(4)\qquad |\cos\tfrac{1}{2}p\pi|\,X^p\leqslant p2^{1/2(p-1)}\,(X^{p-1}+1)+1.$$

It follows that, if only $\cos\frac{1}{2}p\pi\neq0$, X cannot exceed a constant A_p and thus the theorem is established for $p\neq3,5,7,\ldots$

It would not be difficult to supply a special proof for these exceptional values, but it is more convenient to use another, more illuminating, argument, which will give us, besides, information about the constants A_p.

7.22. *If the inequality* 7.21(1) *is true for a certain* $p>1$, *it is also true for the complementary exponent* p'; *moreover* $A_p=A_{p'}$.

Let $g(x)$ be any trigonometrical polynomial with $\mathfrak{M}_p[g]\leqslant1$, and $\bar{g}(x)$ the conjugate polynomial. From Parseval's relation we have

$$\int_0^{2\pi}vg(x)\,dx=-\int_0^{2\pi}u\bar{g}(x)\,dx.$$

It is not difficult to see that $\mathfrak{M}_{p'}[v]$ is the upper bound of the expression on the left for all possible g (§ 4.7.2). The expression on the right does not exceed, in absolute value, $\mathfrak{M}_{p'}[u]\,\mathfrak{M}_p[\bar{g}] \leqslant \mathfrak{M}_{p'}[u]\,A_p\,\mathfrak{M}_p[g] \leqslant A_p\,\mathfrak{M}_{p'}[u]$, so that $\mathfrak{M}_{p'}[v] \leqslant A_p\,\mathfrak{M}_{p'}[u]$ and the theorem follows. At the same time, since Theorem 7.21 was established for $1 < p \leqslant 2$, it holds for $p \geqslant 2$, and in particular for $p = 3, 5, \dots$

7.23. Stein's proof. The preceding proof of Theorem 7.21 is due to M. Riesz. An alternative proof, based on a different idea, has been obtained by Stein[1]). We shall reproduce it here since it is very simple and yields a good estimate for the constants A_p. Its main feature is the use of Green's formula

$$(1) \qquad \int_C \frac{\partial w}{\partial r}\, ds = \iint_S \Delta w\, d\sigma.$$

Here S is the circle $\xi^2 + \eta^2 \leqslant r^2$, C its circumference, and w a function of rectangular variables ξ, η, which, together with its first and second derivatives, is continuous in S; $\partial w/\partial r$ denotes the derivative in the direction of the radius vector, and Δw the expression $\partial^2 w/\partial \xi^2 + \partial^2 w/\partial \eta^2$.

As we already know, it is sufficient to prove Theorem 7.21 for the case $1 < p \leqslant 2$, $u(z) > 0$. Consider $u(z)$, $v(z)$, $|F(z)| = (u^2 + v^2)^{1/2}$ as functions of ξ, η. A simple calculation shows that

$$(2) \qquad \Delta u^p = p(p-1)\, u^{p-2}\, |F'|^2, \quad \Delta |F|^p = p^2\, |F|^{p-2}\, |F'|^2,$$

so that, since $p \leqslant 2$, $|f| \geqslant u$, we find $\Delta |F|^p \leqslant p'\, \Delta u^p$. Let $\mathfrak{M}_p^p[u(re^{ix})] = \lambda(r)$, $\mathfrak{M}_p^p[F(re^{ix})] = \mu(r)$. We shall apply the formula (1) to the functions $w = u^p$ and $w = |F|^p$. Since $ds = r\,dx$, the left-hand sides will represent $r\,d\lambda/dr$ and $r\,d\mu/dr$ respectively, and, in virtue of the inequality $\Delta |F|^p \leqslant p' \Delta u^p$, we obtain $\mu'(r) \leqslant p' \lambda'(r)$. Integrating this inequality with respect to r, and taking into account that $\lambda(0) = \mu(0)$, $p' > 1$, we find $\mu(r) \leqslant p' \lambda(r)$. This is just the inequality 7.21(1), with $A_p = p'^{1/p}$, $1 < p \leqslant 2$. If u is no longer positive, the value of A_p is increased by the factor 2. It follows that $A_p \leqslant 2p^{1/p'} < 2p$ for $p \geqslant 2$. For better estimates we refer the reader to the original paper.

[1]) Stein [1].

7.24. Theorem 7.21 ceases to be true when $p=1$, since the sum $\bar{f}(x)$ of $\bar{\mathfrak{S}}[f]$ is not necessarily integrable (§ 5.221). It follows, in particular, that the proper, i. e. the best possible, value of A_p is unbounded when p tends to 1 or to ∞. The place of Theorem 7.21 is taken by two other theorems. We shall prove the first of them by M. Riesz's method, whereas for the second the method developed in the preceding section will be more convenient.

(i) *If $f(x)$ is integrable, so is $|\bar{f}(x)|^p$, for any $0<p<1$. Moreover there is a constant B_p depending only on p and such that* $\mathfrak{M}_p[\bar{f}] \leqslant B_p \mathfrak{M}[f]$, $0<p<1$ [1]).

(ii) *If $|f|\log^+|f|$ is integrable, then $\bar{f}$ is integrable and* $\bar{\mathfrak{S}}[f]=\mathfrak{S}[\bar{f}]$. *There exist two absolute constants A and B such that*

$$(1) \qquad \int_0^{2\pi} |\bar{f}|\,dx \leqslant A\int_0^{2\pi} |f|\log^+|f|\,dx + B.$$

As regards (i) it is, as before, enough to prove that, if $F(z)=u+iv$ is regular for $|z|<1$, then $\mathfrak{M}_p[v] \leqslant B_p\mathfrak{M}_p[u]$. Suppose first that $u>0$. Taking the real parts in 7.21(2), we have, since $|\arg(u+iv)^p| \leqslant \frac{1}{2}p\pi$,

$$\frac{\cos\frac{1}{2}p\pi}{2\pi}\int_0^{2\pi}(u^2+v^2)^{p/2}\,dx \leqslant \left(\frac{1}{2\pi}\int_0^{2\pi} u\,dx\right)^p.$$

This inequality holds à fortiori if we omit the term u^2 on the left, but then we obtain just what we wanted to prove, with $B_p^p=(2\pi)^{1-p}\sec\frac{1}{2}\pi p$. To remove the assumption $u>0$, we proceed as in § 7.21, but, since Minkowski's inequality does not work for $p<1$, we apply the inequality $|\bar{\varphi}|^p=|\bar{\varphi}_1+\bar{\varphi}_2|^p \leqslant |\bar{\varphi}_1|^p+|\bar{\varphi}_2|^p$ (§ 4.13) and the value of B_p is increased by the factor $2^{1/p}$.

To establish (ii) it is again sufficient to prove the inequality (1) with $f,\bar{f}$ replaced by u,v. Suppose first that $u>e$. We verify that $\Delta(u\log u)=|F'(z)|^2/u$, $\Delta|F|=|F'|^2/|F| \leqslant \Delta(u\log u)$. Denoting by $\lambda(r)$ and $\mu(r)$ the integrals of $u\log u$ and $|F|$ over

---

[1]) Kolmogoroff [4]; Littlewood [3], Hardy [9], Tamarkin [1].
[2]) Zygmund [4]; Titchmarsh [3], Littlewood [4], Stein [1]; $\log^+ x$ denotes the function which is equal to $\log x$ for $x>1$ and to 0 elsewhere.

the interval $0 \leqslant x \leqslant 2\pi$, we find that $\mu'(r) \leqslant \lambda'(r)$, and hence $\mu(r) \leqslant \lambda(r)$, since $|F(0)| = u(0) \leqslant u(0) \log u(0)$.

In the general case we proceed as in § 7.21, viz. put $\varphi(x) = \varphi_1(x) + \varphi_2(x) + \varphi_3(x)$, where $\varphi_1 = \text{Max}\{\varphi(x), e\}$, $\varphi_3 = \text{Min}\{\varphi(x), -e\}$, so that $|\varphi_2(x)| \leqslant e$. Since $\mathfrak{A}[\varphi_2(x)] \leqslant \mathfrak{A}_2[\varphi_2(x)] \leqslant \mathfrak{A}_2[\varphi_2(x)] \leqslant e$, (§ 4.15) the inequality (1) follows, with $A = 2$, $B = 2\pi e$.

That $\overline{\mathfrak{S}}[f] = \mathfrak{S}[\bar{f}]$ is a corollary of the relation $\mathfrak{M}[\bar{f} - \bar{s}_n] \to 0$ which will be established in § 7.3 ($\bar{s}_n$ denote the partial sums of $\overline{\mathfrak{S}}[f]$).

7.25. It is important to observe that the integrability of $|f| \log^+ |f|$ is essential for that of $\bar{f}$, and cannot be replaced by anything less stringent. This follows from the following result, which is, in some respects, a converse of Theorem 7.24(ii).

If f is non-negative and $\bar{f}$ integrable, then $f \log^+ f$ is integrable [1].

Suppose, as we may, that $f \geqslant 1$, and let $u(z)$, $v(z)$ denote the Poisson integral of f and the conjugate harmonic function. Putting $F(z) = u + iv$ we consider the integral of the function $z^{-1} F(z) \log F(z)$, taken round the circle $|z| = r$. Applying Cauchy's theorem and taking the real parts on both sides of the equation, we have

$$(1) \qquad \frac{1}{2\pi} \int_0^{2\pi} \left\{ u \log \sqrt{u^2 + v^2} - v \operatorname{arctg} \frac{v}{u} \right\} dx = u(0) \log u(0).$$

In virtue of the inequality $0 \leqslant v \operatorname{arctg}(v/u) \leqslant \frac{1}{2}\pi |v|$, we obtain

$$(2) \qquad \frac{1}{2\pi} \int_0^{2\pi} u \log u \, dx \leqslant \frac{1}{4} \int_0^{2\pi} |v| \, dx + u(0) \log u(0).$$

In § 7.56 (see also § 7.26(iii)) we shall learn that, if $\bar{f}$ is integrable, then $\overline{\mathfrak{S}}[f] = \mathfrak{S}[\bar{f}]$, so that the integral on the right in (2) is bounded and the result follows by an application of Fatou's lemma.

7.26. Integral B. There exist, as we have already mentioned, functions $f \in L$ such that $\bar{f}$ is not integrable. It is interesting to observe that, with a suitable definition of an integral, more general than that of Lebesgue, the function $\bar{f}$ is integrable.

Given any function $f(x)$, $a \leqslant x < b$, we repeat it periodically in the intervals $a + kh \leqslant x < a + (k+1)h$, $k = \pm 1, \pm 2, \dots$, where $h = b - a$. Let $a = x_0 < x_1 < \dots < x_n = b$ be any subdivision of (a, b), ξ_i an arbitrary point from (x_{i-1}, x_i), and $\rho = \text{Max}(x_i - x_{i-1})$. Consider the expression

---

[1] The theorem is due to M. Riesz.

(1) $$I(t)=\sum_{i=1}^{n} f(\xi_i+t)(x_i-x_{i-1}), \qquad 0\leqslant t<b-a,$$

and suppose that there exists a number I with the following property: for any $\varepsilon>0$ we can find a $\delta=\delta(\varepsilon)$ such that $|I(t)-I|<\varepsilon$, except for t belonging to a set of measure less than ε, provided that $\rho\leqslant\delta$ (independently of the values of x_i, ξ_i). We shall say, then, that $f(x)$ is integrable B over (a,b) and that I is the value of the integral [1]). It is easy to grasp the meaning of the above definition if we proceed as follows: besides the function $f(x)$ we consider a whole family of functions $f_t(x)=f(t+x)$, depending on a parameter t, and for each of them we form Riemannian sums. If $f(x)$ is not integrable R, no $f_t(x)$ is, but it may happen that 'on the whole' those sums approach I. If this happens, f is integrable B; we could also say that f is integrable R 'in measure'.

(i) *If f is integrable L over (a,b), it is also integrable B, both integrals having the same value.*

Put $f=f_1+f_2$, and correspondingly $I(t)=I_1(t)+I_2(t)$, where f_1 is continuous and the integral of $|f_2|$ over (a,b) is less than $\frac{1}{3}\varepsilon^2/(b-a)$. The integral of $|I_2(t)|$ over (a,b) is less than $\frac{1}{3}\varepsilon^2$, so that the set T of t where $|I_2(t)|>\frac{1}{3}\varepsilon$ is of measure $<\varepsilon$. If I, I_1, I_2 are the integrals of f, f_1, f_2 over (a,b), then $|I(t)-I|\leqslant|I_1(t)-I_1|+|I_2(t)|+|I_2|$. The first term on the right is less than $\frac{1}{3}\varepsilon$ if only $\rho\leqslant\delta=\delta(\varepsilon)$. The second is less than $\frac{1}{3}\varepsilon$ for $t\,\bar{\epsilon}\,T$. The third is less than $\frac{1}{3}\varepsilon^2(b-a)<\frac{1}{3}\varepsilon$, assuming, as we may, that $\varepsilon(b-a)<1$. Hence $|I(t)-I|<\varepsilon$ for $t\,\bar{\epsilon}\,T$, $|T|<\varepsilon$, if only $\rho\leqslant\delta$, and the theorem follows.

(ii) *For every $f\in L$, $\overline{f}$ is integrable B over $(0,2\pi)$, and $\overline{\mathfrak{S}}[f]=\mathfrak{S}[\overline{f}]$* [2]).

Substituting $\overline{f}$ for f in the expression (1), we obtain a function $\overline{I}(t)$, conjugate to $I(t)$. By Theorem 7.24(i), we have $\mathfrak{M}_{1/2}[\overline{I}(t)]\leqslant B_{1/2}\mathfrak{M}[I(t)]\leqslant 2\pi B_{1/2}\mathfrak{M}[f]$. It follows that $|\overline{I}(t)|<\frac{1}{2}\varepsilon$, for $t\,\bar{\epsilon}\,T$, $|T|<\varepsilon$, if only the integral of $|f|$ over $(0,2\pi)$ is less than $\eta=\eta(\varepsilon)$. In the general case we put $f=f_1+f_2$, where f_1 is a trigonometrical polynomial and the integral of $|f_2|$ is less than η. We find then that $|\overline{I}(t)|<\varepsilon$ for $t\,\bar{\epsilon}\,T$, $|T|<\varepsilon$, provided that $\rho\leqslant\delta=\delta(\varepsilon)$. Thus the integral B of $\overline{f}$ over $(0,2\pi)$ exists and has the value 0.

We shall now show that the products $\overline{f}\cos kx$, $\overline{f}\sin kx$ are integrable B over $(0,2\pi)$, to the values $-\pi b_k, \pi a_k$, $k=1,2,\ldots$ We may suppose that $a_0=a_1=\ldots=a_k=b_1=\ldots=b_k=0$. We have then

(2a) $\overline{f\cos kx}=\overline{f}\cos kx$, (2b) $\overline{f\sin kx}=\overline{f}\sin kx$.

This is easy to verify when f is a trigonometrical polynomial. Hence (2)

---

[1]) Integral B is one of several definitions of an integral propounded by Denjoy; see Denjoy [3], Boks [1]. Proposition (i) (see below) belongs also to Denjoy, but the proof of the text, which is much simpler, has been given by Saks.

[2]) Kolmogoroff [5]. The example of the series conjugate to 5.12(2) (or simply that of the odd function equal to $1/x\log(x/2\pi)$ in the interval $0<x\leqslant\pi$) shows that a function may be integrable B over $(-\pi,\pi)$ without being integrable B over $(0,\pi)$.

is true if we replace $f, \bar{f}$ by $\sigma_n, \bar{\sigma}_n$, where $\sigma_n, \bar{\sigma}_n$ denote the $(C, 1)$ means of $\mathfrak{S}[f]$, $\bar{\mathfrak{S}}[f]$ respectively. If $n \to \infty$, then $\overline{\bar{\sigma}_n \cos kx} \to \overline{\bar{f} \cos kx}$ and, to prove (2a), it is sufficient to show that $\overline{(\sigma_{n_i} - f) \cos kx} \to 0$ for a sequence $\{n_i\} \to \infty$. This follows from the relations $\mathfrak{M}_{1/2}[\overline{(\sigma_n - f) \cos kx}] \leqslant B_{1/2} \mathfrak{M}[\sigma_n - f] \to 0$ (§ 4.2). Similarly we prove (2b). The formulae (2) show that the products $\bar{f} \cos kx$ and $\bar{f} \sin kx$ are integrable B over $(0, 2\pi)$, the value of the integrals being 0. This completes the proof of (ii). As a corollary we obtain the following theorem.

(iii) *If $\bar{f}$ is integrable L, then $\bar{\mathfrak{S}}[f]$ is the Fourier-Lebesgue series of $\bar{f}$*[1]*.*

7.3. Mean convergence of Fourier series[2]**.** The theorems on conjugate functions which we proved in the preceding paragraph enable us to obtain some results for the partial sums $s_n, \bar{s}_n$ of $\mathfrak{S}[f], \bar{\mathfrak{S}}[f]$.

(i) *If $f \in L^p$, $p > 1$, then $\mathfrak{M}_p[f - s_n] \to 0$.*

(ii) *If f is integrable, then $\mathfrak{M}_p[f - s_n] \to 0$, $\mathfrak{M}_p[\bar{f} - \bar{s}_n] \to 0$ for every $0 < p < 1$.*

(iii) *If $|f| \log^+ |f|$ is integrable, then $\mathfrak{M}[f - s_n] \to 0$, $\mathfrak{M}[\bar{f} - \bar{s}_n] \to 0$.*

Let $s_n^*, \bar{s}_n^*$ denote the modified partial sums $s_n, \bar{s}_n$ (§ 2.3). Since $s_n - s_n^*$ and $\bar{s}_n - \bar{s}_n^*$ tend uniformly to 0, it is sufficient to prove the theorems for $s_n^*, \bar{s}_n^*$ instead of $s_n, \bar{s}_n$. From the formula

$$s_n^*(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \frac{\sin nt}{2 \operatorname{tg} \frac{1}{2} t} dt,$$

replacing $\sin nt$ by $\sin n(t+x) \cos nx - \cos n(t+x) \sin nx$, we see that $|s_n^*(x)| \leqslant |g_1(x)| + |g_2(x)|$, where g is conjugate to $f(x) \sin nx$, g_2 to $f(x) \cos nx$. Theorem 7.21 and Minkowski's inequality give

$$(1) \qquad \mathfrak{M}_p[s_n^*] \leqslant 2 A_p \mathfrak{M}_p[f],$$

an inequality important in itself. Now put $f = f' + f''$, where f' is a trigonometrical polynomial and $\mathfrak{M}_p[f''] < \varepsilon$. Similarly we have $s_n^* = s_n^{*\prime} + s_n^{*\prime\prime}$, $f - s_n^* = (f' - s_n^{*\prime}) + (f'' - s_n^{*\prime\prime})$ and so, if $p > 1$,

$$\mathfrak{M}_p[f - s_n^*] \leqslant \mathfrak{M}_p[f' - s_n^{*\prime}] + \mathfrak{M}_p[f''] + \mathfrak{M}_p[s_n^{*\prime\prime}] = \mathfrak{M}_p[f''] + \mathfrak{M}_p[s_n^{*\prime\prime}]$$

for n large. By (1), the right-hand side does not exceed $(2A_p + 1)\varepsilon$, and the first part of the theorem follows.

If $|f| \log^+ |f|$ is integrable, then

---

[1] See also Titchmarsh [4], Smirnoff [1].

[2] See the papers referred to in the preceding paragraph.

(2) $$\mathfrak{M}[s_n^*] \leqslant 2A\int_0^{2\pi}|f|\log^+|f|\,dx + 2B$$

(§ 7.24). Let us apply this result to the function kf, where k is a positive constant. It follows that

$$\mathfrak{M}[s_n^*] \leqslant 2A\int_0^{2\pi}|f|\log^+|kf|\,dx + \frac{2B}{k} < \varepsilon,$$

if $2B/k = \frac{1}{2}\varepsilon$ and the integral of $2A|f|\log^+|kf|$ over $(0, 2\pi)$ does not exceed $\frac{1}{2}\varepsilon$. To obtain that $\mathfrak{M}[f - s_n^*] \to 0$, we again write $f = f' + f''$, where f' is a polynomial, and the integral of $|f''|\log^+|kf''|$ is small, and proceed as before.

From the formula defining $\bar{s}_n^*$, we conclude that $|\bar{s}_n^*(x) - \bar{f}(x)| \leqslant |g_1(x)| + |g_2(x)|$, g_1 and g_2 having the previous meaning. Thus $\mathfrak{M}[\bar{s}_n^*]$ satisfies an inequality analogous to (2), with $2A$, $2B$ replaced by $3A$, $3B$, and again $\mathfrak{M}[\bar{f} - \bar{s}_n^*] \to 0$.

Theorem (iii) is proved in the same way, except that for $p < 1$ we use the inequality $\mathfrak{M}_p^p[f' + f''] \leqslant \mathfrak{M}_p^p[f'] + \mathfrak{M}_p^p[f'']$.

As corollaries of the above theorems we obtain the following results, the first of which is a generalization of Theorem 4.41(ii).

(iv) *If the Fourier coefficients of a function $f \in L^p$, $p > 1$, are a_n, b_n, those of a $g \in L^{p'}$ are a_n', b_n', we have the Parseval formula*

(3) $$\frac{1}{\pi}\int_0^{2\pi} fg\,dx = \tfrac{1}{2}a_0a_0' + \sum_{n=1}^{\infty}(a_na_n' + b_nb_n'),$$

the series on the right being convergent

(v) *The formula* (3) *holds also if $|f|\log^+|f|$ is integrable and g bounded.*

The proofs are similar to those of Theorems 4.41(ii) and 4.41(iii), if we take into account that $\mathfrak{M}_p[f - s_n] \to 0$ in case (iv) and $\mathfrak{M}[f - s_n] \to 0$ in case (v).

(vi) *For any integrable f there is a sequence of indices n_k such that $s_{n_k}(x)$ converges almost everywhere to $f(x)$; similarly we can find a sequence $\{m_k\}$ such that $\bar{s}_{m_k}(x)$ tends almost everywhere to $\bar{f}(x)$.* This follows from (ii) and Theorem 4.2(ii).

We add that for $\{n_k\}$ and $\{m_k\}$ we may take any sequences increasing sufficiently rapidly and, consequently, $\{n_k\}$ and $\{m_k\}$

may be subsequences of arbitrary sequences of integers tending to $+\infty$.

7.31. Theorem 7.3(i) ceases to be true for $p=1$ or $p=\infty$: $\mathfrak{M}[f-s_n]$ does not necessarily tend to 0 for f integrable, nor does s_n tend uniformly to f for f continuous. It is interesting to observe that if f and $\bar{f}$ are both integrable, or both continuous, then $\mathfrak{S}[f]$ and $\bar{\mathfrak{S}}[f]$ behave much in the same way, as is seen from the following theorems [1]).

(i) *If f and $\bar{f}$ are both continuous, and $\mathfrak{S}[f]$ converges uniformly, so does $\bar{\mathfrak{S}}[f]$. If $\bar{f}$ and f are both bounded and $\mathfrak{S}[f]$ has partial sums uniformly bounded, so has $\bar{\mathfrak{S}}[f]$.*

(ii) *If $\bar{\mathfrak{S}}[f]$ is a Fourier series and $\mathfrak{M}[s_n]$ is bounded, so is $\mathfrak{M}[\bar{s}_n]$; and if $\mathfrak{M}[f-s_n]\to 0$, so does $\mathfrak{M}[\bar{f}-\bar{s}_n]$.*

The proofs are based on the following two lemmas, the first of which may be considered as the limiting case, for $p=\infty$, of the second [2]).

(a) *If $t_n(x)$ is a trigonometrical polynomial of order n, and $|t_n(x)|\leqslant M$, then $|t'_n(x)|\leqslant 2nM$.*

(b) *If $\mathfrak{M}_p[t_n(x)]\leqslant M$, $p\geqslant 1$, then $\mathfrak{M}_p[t'_n(x)]\leqslant 2nM$.*

The proofs are very simple. In the formula

$$t'_n(x)=\frac{1}{\pi}\int_0^{2\pi} t_n(x+u)\,[\sin u+2\sin 2u+\ldots+n\sin nu]\,du$$

we add to the expression in brackets the sum $(n-1)\sin(n+1)u+$ $+(n-2)\sin(n+2)u+\ldots+\sin(2n-1)u$, which, since t_n is a polynomial of order n, does not change the value of the integral. Adding together the terms $k\sin ku$ and $k\sin(2n-k)u$, we obtain the formula

$$(1)\qquad t'_n(x)=\frac{2}{\pi}\int_0^{2\pi} t_n(x+u)\sin nu\,K_{n-1}(u)\,du,$$

K_{n-1} denoting the Fejér kernel. It follows that $|t'_n(x)|$ does not exceed the $(n-1)$-st Fejér mean of the function $2|t_n(x)|$, and it remains to appeal to Theorem 3.22 and the formula 4.33(3).

---

1) Fejér [6], Zygmund [9].

2) The first is due to S. Bernstein [1]. For the second see Zygmund [9] and F. Riesz [3]. The factor 2 on the right may be made to disappear, but this makes no difference to us.

Let σ_n and $\bar{\sigma}_n$ denote the first arithmetic means of $\mathfrak{S}[f]$ and $\bar{\mathfrak{S}}[f]$. Suppose that $\mathfrak{S}[f]$ converges uniformly. The formula 3.13(1) for the difference $\bar{s}_n - \bar{\sigma}_n$ now takes the form: $\bar{\sigma}_n - \bar{s}_n = = s'_n/(n+1) = (s'_n - s'_{n_0})/(n+1) + s'_{n_0}/(n+1)$, where n_0 is fixed and so large that $|s_n - s_{n_0}| < {}^1/_4\,\varepsilon$, uniformly in x, for any $n > n_0$. From (a) we see that $|s'_n - s'_{n_0}| \leqslant \frac{1}{2}\varepsilon n$. Since for $n > n_1 > n_0$ we have $|s'_{n_0}|/(n+1) < \frac{1}{2}\varepsilon$, it follows that $|\bar{\sigma}_n - \bar{s}_n| < \varepsilon$ for $n > n_1$, i. e. $\bar{\sigma}_n - \bar{s}_n \to 0$. But, $\bar{f}$ being continuous, we have $\bar{\sigma}_n \to \bar{f}$, and so $\bar{s}_n \to \bar{f}$, uniformly in x. This gives the first part of (i). The proof of the second part is still simpler and may be left to the reader.

We prove (ii) by the same method, using (b) for $p = 1$.

Considering, for example, the second part of (ii), we observe that $\mathfrak{M}[s_n - s_{n_0}] < {}^1/_4\,\varepsilon$ if n_0 is large enough and $n > n_0$. Thence, arguing as before, we obtain that $\mathfrak{M}[\bar{\sigma}_n - \bar{s}_n] = \mathfrak{M}[s'_n/(n+1)] \to 0$. This and the relation $\mathfrak{M}[\bar{f} - \bar{\sigma}_n] \to 0$, give $\mathfrak{M}[\bar{f} - \bar{s}_n] \to 0$, and the theorem is established.

We shall complete (i) by the following remark. The relation $\bar{\sigma}_n - \bar{s}_n \to 0$ was established under the sole hypothesis that $\mathfrak{S}[f]$ converges uniformly. We have then $\bar{\mathfrak{S}}[f] = \mathfrak{S}[\bar{f}]$, where $\bar{f} \in L^2$, and so $\bar{\sigma}_n$ converges almost everywhere. Therefore[1]), *if* $\mathfrak{S}[f]$ *converges uniformly,* $\bar{\mathfrak{S}}[f]$ *converges almost everywhere. If the partials sums of* $\mathfrak{S}[f]$ *are uniformly bounded, the partial sums of* $\bar{\mathfrak{S}}[f]$ *are bounded at almost every point.*

7.4. Privaloff's theorem. Theorem 7.21 teaches us that, except in limiting cases, the functions f and $\bar{f}$ have, so to speak, the same integrability. It is therefore natural to ask if anything similar is true for continuity. The answer is given by the following theorem due to Privaloff.

If $f \in \mathrm{Lip}\,\alpha$, $0 < \alpha < 1$, *then* $\bar{f} \in \mathrm{Lip}\,\alpha$[2]).

Consider the formulae

$$\bar{f}(x) = -\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{f(x+t) - f(x)}{2\,\mathrm{tg}\,\frac{1}{2}t}\,dt, \tag{1}$$

$$\bar{f}(x+h) = -\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{f(x+t) - f(x+h)}{2\,\mathrm{tg}\,\frac{1}{2}(t-h)}\,dt,$$

[1]) Fejer [6]; see also Privaloff [4], Zygmund [7].
[2]) Privaloff [3].

where $h > 0$. They differ slightly from 7.1(1), but, since $\operatorname{tg}\frac{1}{2}t$ is an odd function of t, the additional terms vanish. The integrands are $O(|t|^{\alpha-1})$, $O(|t-h|^{\alpha-1})$ respectively. Consequently, if we cut the interval $(-2h, 2h)$ out of the interval of integration $(-\pi, \pi)$ in (1), we commit an error $R_1 = O(h^\alpha)$ in the first formula and an error $R_2 = O(h^\alpha)$ in the second. Hence the difference $\bar{f}(x+h) - \bar{f}(x)$ is equal to

$$(2) \quad -\frac{1}{\pi}\left(\int_{-\pi}^{-2h} + \int_{2h}^{\pi}\right)[f(x+t) - f(x)]\,[\operatorname{ctg}\tfrac{1}{2}(t-h) - \operatorname{ctg}\tfrac{1}{2}t]\,dt + \\ + R_2 - R_1 + R,$$

where $R = [f(x+h) - f(x)]\int_{2h}^{\pi}[\operatorname{ctg}\frac{1}{2}(t-h) - \operatorname{ctg}\frac{1}{2}(t+h)]\,dt =$

$$= O(h^\alpha)\left[2\log\frac{\sin\frac{1}{2}(t-h)}{\sin\frac{1}{2}(t+h)}\right]_{2h}^{\pi} = O(h^\alpha).$$

Since $\operatorname{ctg}\frac{1}{2}(t-h) - \operatorname{ctg}\frac{1}{2}t = \sin\frac{1}{2}h/\sin\frac{1}{2}(t-h)\sin\frac{1}{2}t$, the function under the integral sign in (2) is $O(|t|^\alpha)\cdot O(h|t|^{-2})$, hence the integral itself is $O(h^\alpha)$. Collecting the terms, we find that $\bar{f}(x+h) - \bar{f}(x) = O(h^\alpha)$ uniformly in x, and the theorem is established.

The theorem fails for $\alpha = 0$ and $\alpha = 1$. The function conjugate to $\sin x + \frac{1}{2}\sin 2x + \ldots = \frac{1}{2}(\pi - x)$, $0 < x < 2\pi$, is not bounded. Integrating the last series formally, we obtain a function which is Lip 1, and whose conjugate is not. Repeating the previous argument we find that, *if* $f \in$ Lip 1, *then* $\omega(\delta; \bar{f}) = O(\delta \log 1/\delta)$.

7.5. Power series of bounded variation. We conclude this chapter by a few theorems on Fourier series of functions which, together with their conjugate, are of bounded variation. It will be more convenient to state these theorems in the form bearing on power series. We shall say that a power series

$$(1) \qquad a_0 + a_1 z + a_2 z^2 + \ldots = F(z)$$

is of bounded variation, if its real and imaginary components, for $z = e^{ix}$, are both Fourier series of functions of bounded variation. We know (§ 2.631) that $F(e^{ix})$ is then continuous; consequently the series (1) converges uniformly for $|z| = 1$, and hence converges uniformly for $|z| \leqslant 1$. The theorems we aim at are as follows.

(i) *If the power series* (1) *is of bounded variation, it converges absolutely on the circle* $|z| = 1$ [1].

(ii) *If the power series* (1) *is of bounded variation, the function* $F(e^{ix})$ *is absolutely continuous* [2].

We shall base the proofs on a number of lemmas which are interesting and important in themselves.

7.51. A function $F(z)$, regular for $|z| < 1$, is said to belong to the class H^p, $p > 0$, if the expression

$$\mu_p(r) = \mu_p(r; F) = \frac{1}{2\pi}\int_0^{2\pi} |F(re^{ix})|^p \, dx$$

is bounded as $r \to 1$ [3]. We shall write H instead of H^1, and μ instead of μ_1. If $p > 1$, H^p coincides with the class of power series whose real parts are Poisson's integrals of functions belonging to L^p. The real and imaginary parts of a function belonging to H are represented by Poisson-Stieltjes integrals (§ 4.36).

In virtue of Theorems 2.13 and 4.36(ii), a necessary and sufficient condition that the series 7.5(1) should be of bounded variation is that the function $G(z) = zF'(z) = a_1 z + 2a_2 z^2 + \ldots$ should belong to H. It is familiar that $2\pi\mu(r; zF')$ represents the length of the curve $w = F(z)$, $|z| = r$.

The first lemma we need is as follows.

If $G(z) = G_1(z)\, G_2(z) = \alpha_0 + \alpha_1 z + \ldots$, *and* $\mu_2(r; G_1) \leqslant A_1^2$, $\mu_2(r; G_2) \leqslant A_2^2$, *where* $A_1 \geqslant 0$, $A_2 \geqslant 0$, *the series* $|\alpha_1| + |\alpha_2|/2 + |\alpha_n|/n + \ldots$ *converges to a sum* $\leqslant \pi A_1 A_2$.

Put $G_k(z) = \alpha_0^{(k)} + \alpha_1^{(k)} z + \ldots$, $G_k^*(z) = |\alpha_0^{(k)}| + |\alpha_1^{(k)}| z + \ldots$, $k = 1, 2$, $G^*(z) = G_1^*(z)\, G_2^*(z) = \alpha_0^* + \alpha_1^* z + \ldots$ In virtue of Parseval's relation we have $\mu_2(r; G_k) = \mu_2(r; G_k^*)$, and it is easy to see that $|\alpha_n| \leqslant \alpha_n^*$, $n = 0, 1, \ldots$ Moreover, by Schwarz's inequality, we have $\mu(r; G^*) \leqslant \mu_2^{1/2}(r; G_1^*)\, \mu_2^{1/2}(r; G_2^*) = \mu_2^{1/2}(r; G_1)\, \mu_2^{1/2}(r; G_2) \leqslant A_1 A_2$.

Let us fix a value of r and consider the absolutely convergent series $\alpha_1^* r \sin x + \alpha_2^* r^2 \sin 2x + \ldots = \Im\{G^*(re^{ix})\}$. Multiplying both sides of the equation by $\frac{1}{2}(\pi - x)$, integrating the result

[1] Hardy and Littlewood [10]. See also Fejér [9].

[2] F. and M. Riesz [1].

[3] Hardy [10].

over $(0, 2\pi)$, and taking into account that then n-th sine coefficient of $\frac{1}{2}(\pi - x)$ is $1/n$, we obtain

$$(1) \quad \sum_{n=1}^{\infty} \frac{\alpha_n^*}{n} r^n = \frac{1}{\pi}\int_0^{2\pi} \Im\, G^*(re^{ix})\, \tfrac{1}{2}(\pi - x)\, dx \leqslant \frac{1}{2}\int_0^{2\pi} |G^*(re^{ix})|\, dx.$$

The last integral does not exceed $\pi A_1 A_2$. Making $r \to 1$, we find that $\alpha_1^* + \frac{1}{2}\alpha_2^* + \ldots \leqslant \pi A_1 A_2$, and, since $|\alpha_n| \leqslant \alpha_n^*$, the lemma follows.

7.52. In virtue of this lemma, to prove Theorem 7.5(i) it would be sufficient to show that the function $G(z) = zF'(z) = a_1 z + 2a_2 z^2 + \ldots$ is a product of two functions G_1, G_2 of the class H^2. This proposition will be established later, but for our actual purposes a less strong result will do. Suppose namely that $G(z)$ has only a finite number of zeros $\zeta_1, \zeta_2, \ldots \zeta_k$ in the circle $|z| < 1$. Put $b_h(z) = z$ if $\zeta_h = 0$; if $\zeta_h \neq 0$, let $b_h(z) = (z - \zeta_k)/(1 - z\bar{\zeta}_k)$, $B(z) = b_1(z)\, b_2(z) \ldots b_k(z)$. Each function $b_h(z)$ is regular for $|z| \leqslant 1$, has a simple zero at ζ_k and only there, and $|b_h(z)| = 1$ for $|z| = 1$ [1]. Therefore the function $H(z) = G(z)/B(z)$ is regular for $|z| < 1$, and, as $r \to 1$, $\overline{\lim}\, \mu(r; H) = \overline{\lim}\, \mu(r; G)$.

Let $A = \overline{\lim}\, \mu(r; G)$. The function $H(z)$ has no zeros for $|z| < 1$, and so $\sqrt{H(z)}$ is regular for $|z| < 1$. Put $G_1(z) = \sqrt{H(z)}$, $G_2(z) = \sqrt{H(z)}\, B(z)$, so that $G_1 G_2 = G$. It follows that $\mu_2[r; G_1] = \mu[r; H]$, $\mu_2[r; G_2] \leqslant \mu[r; H]$; $\overline{\lim}\, \mu_2[r; G_k] \leqslant A$ as $r \to 1$, $k = 1, 2$. Now, as it is seen from Parseval's relation, $\mu_2(r)$ increases with r, so that we have $\mu_2(r; G_k) \leqslant A$ for $r < 1$. An appeal to the lemma of § 7.51 gives the following result. *If $zF'(z)$ has only a finite number of zeros in $|z| < 1$, and if $\overline{\lim}\, \mu(r; zF') \leqslant A$, then $|a_1| + |a_2| + \ldots \leqslant \pi A$.*

Now it is easy to complete the proof of Theorem 7.5(i). Let μ now denote the upper bound of $\mu(r, zF')$ for $0 \leqslant r < 1$. (It will be proved in § 7.53(i) that $\mu(r)$ is a non-decreasing function of r, so that $\mu = A$, but this result is not required here). If $0 < \rho < 1$, the function $\rho z F'(z\rho)$ has only a finite number of zeros

---

[1] This last fact, familiar to anyone acquainted with the elements of conformal representation, may be proved as follows: $|b_h(e^{ix})| = |e^{ix} - \zeta_h| / |1 - e^{ix}\bar{\zeta}_h| = |e^{ix} - \zeta_h| / |e^{-ix} - \bar{\zeta}_h| = |e^{ix} - \zeta_h| / |\overline{e^{ix} - \zeta_h}| = 1$. It follows that $|b_h(z)| < 1$ for $|z| < 1$.

for $|z|<1$. Thus $|a_1|\rho+|a_2|\rho^2+\ldots\leqslant\pi\mu$; making $\rho\to 1$, we find that $|a_1|+|a_2|+\ldots\leqslant\pi\mu$ and the theorem follows.

7.521. As a corollary of Theorems 7.5(i) and 7.24(ii) we obtain: *If $F(x)$ is absolutely continuous and $|F'(x)|\log^+|F'(x)|\in L$, then $\mathfrak{S}[F]$ converges absolutely* (§ 6.36).

7.53. Passing to the proof of Theorem 7.5(ii), we shall again require a few lemmas

(i) *If $F(z)$ is regular for $|z|<1$, $\mu_p(r;F)$ is a non-decreasing function of r.* It is not difficult to deduce this from the following proposition which we shall prove first.

(ii) *If $f_1(z), f_2(z), \ldots, f_n(z)$ are regular inside and on the boundary of a plane region R, and $\varphi(z)=|f_1(z)|^p+\ldots+|f_n(z)|^p$, $p>0$, the function $\varphi(z)$ cannot attain a proper maximum inside B.*

Suppose, on the contrary, that $\varphi(z)$ does attain such a maximum at a point z_0 interior to R. Let C be a circle $|z-z_0|\leqslant r$ contained in R and such that (a) if $f_k(z_0)\neq 0$, then $f_k(z)\neq 0$ in C, $k=1,2,\ldots n$, (b) at a point z_1, $|z_1-z_0|=r$, $\varphi(z)$ takes a value smaller than $\varphi(z_0)$. Let $\psi(z)$ be the sum of terms $\varepsilon_k f_k^p(z)$ extended over the values of k for which $f_k(z_0)\neq 0$. The unit factors ε_k are so chosen that the function $\psi(z)$, which is regular in C, takes the value $\varphi(z_0)$ at the point z_0. For every z, $|z-z_0|\leqslant r$, we have

$$|\psi(z)|\leqslant|f_1(z)|^p+\ldots+|f_n(z)|^p=\varphi(z)\leqslant\varphi(z_0)=\psi(z_0)=|\psi(z_0)|,$$

and for $z=z_1$ we actually have $\varphi(z)<\varphi(z_0)$, i. e. $|\psi(z)|<|\psi(z_0)|$. This is in contradiction with the principle of maximum and (ii) is established.

Consider now the function $\varphi_n(z)=\{|F(\eta_1 z)|^p+\ldots+|F(\eta_n z)|^p\}/n$, where $\eta_1, \eta_2, \ldots, \eta_n$ are the n-th unit roots. It is obvious that, for every $0<r<1$, $\varphi_n(re^{ix})\to\mu_p(r;F)$ uniformly in x. Let $0<\rho<r<1$ and let $\operatorname{Max}|\varphi_n(z)|$ for $|z|\leqslant r$ be attained at a point $z=re^{ix}$. We have then $\varphi_n(\rho e^{ix})\leqslant\varphi_n(re^{ix})$, and, making $n\to\infty$, $\mu(\rho)\leqslant\mu(r)$.

(iii) *Let $\zeta_1, \zeta_2, \ldots$ be a sequence of points such that $0<|\zeta_n|<1$, and that the product $|\zeta_1|.|\zeta_2|\ldots$ converges. If $\zeta_n^*=1/\bar{\zeta}_n$, the product*

$$\prod_{n=1}^{\infty}\frac{z-\zeta_n}{z-\zeta_n^*}\cdot\frac{1}{|\zeta_n|} \tag{1}$$

converges absolutely and uniformly in every circle $|z|\leqslant r<1$, to a function $B(z)$ vanishing at the points ζ_n and only there.

The terms of (1) differ only by constant unit factors from the expressions $b_h(z)$ considered in § 7.52. If $|z| \leqslant r$, the difference $1-(z-\zeta_n)/(z-\zeta_n^*)=(\zeta_n-\zeta_n^*)/(z-\zeta_n^*)$ does not exceed $(1-|\zeta_n|^2)/(1-r)<2(1-|\zeta_n|)/(1-r)$ in absolute value; and since, by hypothesis, the series $(1-|\zeta_1|)+(1-|\zeta_2|)+\dots$ converges, the product with factors $(z-\zeta_n)/(z-\zeta_n^*)$ converges absolutely and uniformly for $|z| \leqslant r$. So does the product (1). Since the terms of (1) are less than 1 in absolute value, we obtain that $|B(z)|<1$ for $|z|<1$ and the lemma is established.

(iv) *If $\zeta_1, \zeta_2, \dots$ are all the zeros, different from the origin, of a function $F(z) \in H^p$, $|z|<1$, each counted according to its multiplicity, the product $|\zeta_1| \cdot |\zeta_2| \dots$ converges* [1]. Let $B_n(z)$ denote the n-th partial product of (1) multiplied by z^k, if $F(z)$ has a zero of order k at the origin. The relation $\mu_p(r; F) \to \mu$ as $r \to 1$ implies $\mu_p(r; F/B_n) \to \mu$, $(n=1, 2, \dots)$ and so, by (i), $\mu_p(r; F/B_n) \leqslant \mu$. Making $r=0$ we find $|\zeta_1 \cdot \zeta_2 \dots \zeta_n| \geqslant \mu^{-1/p} |F(z)/z^k|_{z=0}$ and the lemma follows.

(v) *If $\mu_p(r; F) \leqslant \mu$, $0 \leqslant r<1$, we have $F(z)=G(z)B(z)$, where $|B(z)| \leqslant 1$, $G(z)$ is regular and different from 0, and $\mu_p(r; G) \leqslant \mu$* [2].

This lemma, which is fundamental for the whole theory, now follows immediately. If $F(z) \neq 0$ for $|z|<1$, we may put $B(z)=1$, $G(z)=F(z)$. If $\zeta_1, \zeta_2, \dots, B_n(z)$ have the same meaning as in (iv), we put $B(z)=\lim B_n(z)$. From the formula $\mu_p(r; F/B_n) \leqslant \mu$, we deduce that $\mu_p(r; G) \leqslant \mu$, where the function $G=F/B$ has no zero for $|z|<1$. Since $|B|<1$, the lemma is established.

(vi) *If $F \in H$, then $F=F_1 F_2$ with F_1 and F_2 belonging to H^2.* If $F=GB$, where G and B have the same meaning as in (v), we put $F_1=\sqrt{G}$, $F_2=\sqrt{G}B$. Since $\mu_2(r; F_k) \leqslant \mu(r; G)$, $k=1, 2$, the lemma follows.

7.55. Now we are in a position to prove Theorem 7.5(ii), which we state in the following equivalent form. *If the power*

---

[1]) Lemma (iv), as well as some other results of this section, is known to be true for a more general class of functions, viz. for functions F such that $\mathfrak{M}[\log^+ F(re^{ix})]=O(1)$. The latter class, although very important in the general theory of analytic functions, has less applications to the theory of trigonometrical series.

[2]) F. Riesz [4].

series 7.5(1) *belongs to* H, *the real and imaginary parts of the series on the unit circle are Fourier series.* It is sufficient to show that $\mathfrak{M}[F(re^{ix})-F(\rho e^{ix})]\to 0$ as $r,\rho\to 1$ (§ 4.36). Using the last lemma of the previous section and applying Schwarz's inequality, we easily obtain

$$\mathfrak{M}[F(re^{ix})-F(\rho e^{ix})]\leqslant \mathfrak{M}_2[F_1(re^{ix})]\,\mathfrak{M}_2[F_2(re^{ix})-F_2(\rho e^{ix})]+$$

$$+\mathfrak{M}_2[F_2(\rho e^{ix})]\,\mathfrak{M}_2[F_1(re^{ix})-F_1(\rho e^{ix})].$$

Since the second factor in each term on the right tends to 0 as $r,\rho\to 1$, the result follows.

7.56. From the lemmas established in the preceding sections we shall deduce a number of interesting consequences.

(i) *If* $F(z)\in H^p$, *then, for almost every* $z_0=e^{ix_0}$, $F(e^{ix_0})=\lim F(z)$ *exists and is finite as* $z\to z_0$ *along any path not touching the circle* [1]. This theorem is only novel in the case $p<1$. With the notation of § 7.54(v) put $F_1(z)=G^{p/2}(z)$, $F_2(z)=B(z)$. F_1 and F_2 belong to H^2. Since for each of them our theorem is true, it is also true for $F=F_1^{2/p}F_2$.

(ii) *The function* $|F(e^{ix})|^p$ *of* (i) *is integrable.* This is a consequence of Fatou's lemma.

(iii) *If* $F(z)\in H^p$, *then* $\mathfrak{M}_p[F(re^{ix})-F(e^{ix})]\to 0$ *as* $r\to 1$ [2]. This theorem is known to us for $p>1$ (§ 4.36). Let $p\leqslant 1$, $0<r<\rho<1$. If F_1 and F_2 have the same meaning as in (i), then, applying the first inequality of 4.13(3), we obtain

$$|F(re^{ix})-F(\rho e^{ix})|^p\leqslant |F_1(\rho e^{ix})|^2\,|F_2(re^{ix})-F_2(\rho e^{ix})|^p+$$

$$+|F_2(re^{ix})|^p\,|F_1^{2/p}(re^{ix})-F_1^{2/p}(\rho e^{ix})|^p.$$

Making $\rho\to 1$ and integrating over $(0,2\pi)$, we find

---

[1]) F. Riesz [4]. The theorem is false for harmonic functions: there is a harmonic function $u(z)$, $|z|<1$, such that $\mu_p(r;u)=O(1)$ for every $0<p<1$, while $\lim\limits_{r\to 1}u(re^{ix})$ exists only in a set of measure 0. See Hardy and Littlewood [12].

[2]) F. Riesz [4].

$$\int_0^{2\pi} |F(re^{ix}) - F(e^{ix})|^p dx \leqslant \int_0^{2\pi} |F_1(e^{ix})|^2 |F_2(re^{ix}) - F_2(e^{ix})|^p dx +$$
$$+ \int_0^{2\pi} |F_2(re^{ix})|^p |F_1^{2/p}(re^{ix}) - F_1^{2/p}(e^{ix})|^p dx.$$

The first integral on the right tends to 0 with $1-r$ since the product $|F_1(e^{ix})|^2 |F_2(re^{ix}) - F_2(e^{ix})|^p$ is less than the integrable function $2^p |F_1(e^{ix})|^2$ and tends to 0 almost everywhere. Let $F_1^{1/p}(z) = L(z)$; $L(z) \in H^{2p}$. Since $|F_2| \leqslant 1$, the second integral does not exceed

$$(1) \quad \left[\int_0^{2\pi} |L(re^{ix}) - L(e^{ix})|^{2p} dx\right]^{1/2} \left[\int_0^{2\pi} |L(re^{ix}) + L(e^{ix})|^{2p} dx\right]^{1/2}.$$

The first factor here tends to 0 if $2p > 1$, the second is bounded, and the result follows for $p > 1/2$. Assuming this, we obtain, from (1), the result for $p > 1/4$, and so on.

(iv) *If* $F(z) \in H^\alpha$, *and* $|F(e^{ix})|^\beta$ *is integrable for* $\beta > \alpha$, *then* $F(z) \in H^\beta$ [1]. The theorem is obvious if $\alpha > 1$. It is also simple if $F(z) \neq 0$ for $|z| < 1$; for if $G(z) = F^{\alpha/2}(z)$, then $G(z) \in H^2$ and $G(e^{ix}) \in L^{2\beta/\alpha}$ so that $G(z) \in H^{2\beta/\alpha}$, $F(z) \in H^\beta$.

In the general case we have $F = GB$, where $G(z) \neq 0$, $G \in H^\alpha$, and the function B is a product of certain rational functions (§ 7.54(v)). Since $|B(z)| < 1$, the function $B(e^{ix})$ exists for almost every x and $|B(e^{ix})| \leqslant 1$. We shall show that $|B(e^{ix})| = 1$ for almost all x. Taking this result for granted, we can easily prove our theorem. For if $F(e^{ix}) \in L^\beta$, $|B(e^{ix})| = 1$, then $G(e^{ix}) \in L^\beta$ and, since $G(z) \in H^\alpha$, $G(z) \neq 0$, we obtain that $G(z) \in H^\beta$, in virtue of the case already dealt with. Since $F(z) = B(z) H(z)$, $F(z) \in H^\beta$ and the theorem is established.

Using Theorem 7.24(i), we obtain, as a corollary, the following proposition.

(v) *If the function* $\bar{f}$ *conjugate to an integrable function* f *is integrable, then* $\bar{\mathfrak{S}}[f] = \mathfrak{S}[\bar{f}]$.

We have still to prove that $|B(e^{ix})| = 1$ for almost every x. We may obviously assume that the number of zeros $\zeta_1, \zeta_2, \ldots$ is infin-

---

[1] Smirnoff [1].

ite and that $F(0) \neq 0$. Since $|B(z)| \leqslant 1$, it is sufficient to show that $\mu(r; B) \to 1$ as $r \to 1$. Now $\mu(0; B) = |\zeta_1| \cdot |\zeta_2| \ldots$ and, since $\mu(r)$ is a non-decreasing function of r, $\lim \mu(r; B) \geqslant |\zeta_1| \cdot |\zeta_2| \ldots$ Let B_N denote the N-th partial product of 7.53(1) and R_N the product of the remaining terms, so that $B = B_N R_N$. Then we have $\lim\limits_{r \to 1} \mu(r; R_N) \geqslant |\zeta_{N+1}| |\zeta_{N+2}| \ldots$ and, since $|B_N(z)|$ tends uniformly to 1 as $|z| \to 1$, we obtain that $\lim \mu(r; B) \geqslant |\zeta_{N+1}| |\zeta_{N+2}| \ldots$ Taking N arbitrarily large, we see that $\lim \mu(r; B) \geqslant 1$, i. e. $\lim \mu(r; B) = 1$.

7.6. Miscellaneous theorems and examples.

1. The formula 7.1(1) may be written $\overline{f}(x) = -\frac{1}{\pi}\int_0^\infty \frac{f(x+t) - f(x-t)}{t}\, dt$ (§ 2.9.8).

2. There is an integrable $f(x)$ such that $\overline{f}(x)$ is non-integrable in every interval. Lusin [1].

[Take $f \geqslant 1$ such that $f \log f$ is nowhere integrable, and apply Theorems 7.25, 2.531].

3. (i) If $|f(x)| \leqslant 1$, then $\exp \lambda |\overline{f}|$ is integrable for every $\lambda < \frac{1}{2}\pi$. (ii) If f is continuous, $\exp \lambda |\overline{f}|$ is integrable for every λ. (iii) If s_n, $\overline{s}_n$ denote the partial sums of $\mathfrak{S}[f]$, $\overline{\mathfrak{S}}[f]$ respectively, then $\mathfrak{M}[\exp \lambda |f - s_n|; 0, 2\pi] \to 2\pi$, $\mathfrak{M}[\exp \lambda |\overline{f} - \overline{s}_n|] \to 2\pi$, for $\lambda < \frac{1}{4}\pi$ if $|f| \leqslant 1$, and for any λ if f is continuous. Zygmund [4]; see also Warschawski [1].

[To prove (i) let F, u, v have the same meaning as in § 7.21. Then

$$\frac{1}{2\pi i}\int_{|z|=r} z^{-1} \exp\{\pm i\lambda F(z)\}\, dz = \exp\{\pm i\lambda F(0)\}, \quad \int_0^{2\pi} \cos \lambda u \, \exp(\pm \lambda v)\, dx = \text{const.}].$$

4. If $F(z) = u(z) + iv(z)$ is an arbitrary function regular for $|z| < 1$ and such that $u > 0$, $v > 0$, then $u(e^{ix}) \in L^{2-\varepsilon}$, $v(e^{ix}) \in L^{2-\varepsilon}$ for every $\varepsilon > 0$ but not necessarily for $\varepsilon = 0$.

[Let $F_1 = F \exp(-\pi i/4) = u_1 + iv_1$, where $|v_1| \leqslant u_1$. Apply to F_1 an argument similar to that of Theorem 7.24(i)].

5. Let Φ, Ψ and Φ_1, Ψ_1 be two pairs of Young's complementary functions. If, for any $f \in L^*_\Phi$, (i) the conjugate function $\overline{f}$ belongs to $L^*_{\Phi_1}$ and (ii) there exists a constant A independent of f and such that $\|\overline{f}\|_{\Phi_1} \leqslant A \|f\|_\Phi$, then, for any $g \in L^*_{\Psi_1}$, we have $\overline{g} \in L^*_\Psi$ and, moreover, $\|g\|_\Psi \leqslant 2A \|g\|_{\Psi_1}$.

[It is sufficient to prove that, if $\|v\|_{\Phi_1} \leqslant A \|u\|_\Phi$ for any function $u + iv$ regular for $|z| < 1$ and such that $v(0) = 0$, then $\|v\|_\Psi \leqslant 2A \|u\|_{\Psi_1}$. Denoting by h an arbitrary polynomial such that $\mathfrak{M}[\Phi|h|; 0, 2\pi] \leqslant 1$, we have

$$\|v\|_\Psi = \operatorname*{Sup}_h \left| \int_0^{2\pi} v\, h\, dx \right| = \operatorname*{Sup}_h \left| \int_0^{2\pi} u\, \overline{h}\, dx \right| \leqslant 2A\sigma \|u\|_{\Psi_1},$$

where $\sigma = \mathrm{Max}\{1, \mathrm{Sup}\, \mathfrak{M}[\Phi_1 | \bar{h}/2A|]\}$ (§ 4.541). On the other hand, since $\mathfrak{M}[\Phi | h|] \leqslant 1$, we have $\| h \|_{\Phi} \leqslant 2$, and so, by (ii), $\| h \|_{\Phi_1} \leqslant A \| h \|_{\Phi} \leqslant 2A$. Hence (§ 4.541) $\mathfrak{M}[\Phi_1 | h/2A|] < 1$, $\sigma = 1$, and $\| v \|_{\Psi} \leqslant 2A \| u \|_{\Psi_1}$.

6. Let $s(x)$, $x \geqslant 0$, be a function which is *concave* (i. e. $-s$ is convex), non-negative, has a continuous derivative for $x > 0$, and tends to $+\infty$ with x, and let $S(x)$ be the indefinite integral of $s(x)$. Let $R(x)$, $x \geqslant 0$, be a function which is non-negative, convex, tends to $+\infty$ with x, and has the first and second derivatives continuous for $x > 0$. Suppose in addition that there is a constant $C > 0$ such that $S''(x) + S'(x)/x \leqslant CR''(x)$. Under these conditions, *if* $f \in L_R$, *then* $\bar{f} \in L_S$.

[The proof is substantially the same as that of § 7.23. Observe that $S(2x) \leqslant C_1 S(x) > 0$ with C_1 independent of x. If $S(x) \leqslant R(x)$, then we have $\mathfrak{M}[S|\bar{f}|] \leqslant C_2 \mathfrak{M}[R|f|]$, where C_2 is independent of f].

7. (i) If $|f|(\log^+|f|)^\alpha \in L$, $\alpha > 0$, then $|\bar{f}| \log^{\alpha-1}(2+|\bar{f}|) \in L$, and there are two constants $A = A_\alpha$, $B = B_\alpha$ such that

$$\int_0^{2\pi} |\bar{f}| \log^{\alpha-1}(2+|\bar{f}|)\, dx \leqslant A \int_0^{2\pi} |f| (\log^+|f|)^\alpha\, dx + B.$$

(ii) If the integral of $\exp|f|^\alpha$, $\alpha > 0$, over $(0, 2\pi)$ is $\leqslant 1$, then the function $\exp \lambda |\bar{f}|^\beta$ is integrable for $\beta = \alpha/(\alpha+1)$ and $\lambda < \lambda_0 = \lambda_0(\alpha)$.

(iii) Theorem (i) is not true for $\alpha = 0$.

8. Let σ_n and $\bar{\sigma}_n$ denote the k-th arithmetic means, $k > 0$, for $\mathfrak{S}[dF]$ and $\bar{\mathfrak{S}}[dF]$ respectively, where F is a function of bounded variation. If $f = F'$, and g denotes the function defined by 7.11(1), then $\mathfrak{M}_p[\sigma_n - f] \to 0$, $\mathfrak{M}_p[\bar{\sigma}_n - g] \to 0$ for every $0 < p < 1$.

9. The constant A_p of Theorem 7.21 satisfies an inequality $A_p > Ap$, where A is a positive absolute constant. Titchmarsh [5].

[Consider the function $f(x) = (\pi - x)/2$, $0 < x < 2\pi$, and observe that $\bar{f}(x) \sim \log 1/x$ as $x \to +0$].

10. Let $P_n(z) = (1+z+z^2+\ldots+z^n)^2/(n+1) = (1+2z+3z^2+\ldots+z^{2n})/(n+1)$, $Q_n(z) = (1+2z+3z^2+\ldots+(n+1)z^n)/(n+1)$. If $\Sigma|\alpha_k| < \infty$, $m_k + 2n_k < m_{k+1}$, $k = 1, 2, \ldots$, the real and imaginary parts of the power series $\Sigma \alpha_k z^{m_k} P_{n_k}(z)$, $z = e^{ix}$, are Fourier series. If in addition $\alpha_k \log n_k \to \infty$, the partial sums t_ν of the power series satisfy the relation $\overline{\lim}\, \mathfrak{M}[t_\nu(e^{ix})] = \infty$. The example is due to F. Riesz; see Zygmund [9].

[The point of this example is that the phenomenon observed in § 5.12 for Fourier series subsists for power series. Use the relations $\mathfrak{M}[P_n(e^{ix})] = 2\pi$, $\mathfrak{M}[Q_n(e^{ix})] > C \log n$, where $C > 0$ is an absolute constant].

11. Let $F(z) = u(z) + iv(z)$ be a function regular for $|z| < 1$. If, for any point $x_0 \in E$, $|E| > 0$, $\lim u(z)$ exists and is finite as $z \to e^{ix_0}$ along any path not touching the circle, the same is true for the function $v(z)$ and almost every point $x_0 \in E$. Privaloff [2]; see also Plessner [3].

For the proof, which is rather deep, the reader is referred to the original papers.

12. If $F(x)$ is integrable and $F'(x)$ exists and is finite for $x \in E$, $|E| > 0$, the integral (*) $-\frac{1}{\pi}\int_0^\pi \frac{F(x+t)+F(x-t)-2F(x)}{4\sin^2 \frac{1}{2}t}\,dt$ exists for almost every $x \in E$. Plessner [3].

[This follows from the previous theorem and Theorem 3.9.13].

13. If the conditions of the previous theorem are satisfied, then, for almost every $x \in E$, $\mathfrak{S}'[F]$ is summable (C, k), $k > 1$, to the value (*).

14. If $f(x)$ is integrable in the sense of Denjoy-Perron, the function $\bar{f}(x)$ defined by 7.1(1) exists for almost every x. Plessner [3].

15. If either (i) $0 < \alpha < 1$, $p \geq 1$, or (ii) $\alpha = 1$, $p > 1$, and if f belongs to Lip(α, p), so does $\bar{f}$. The theorem is false for $\alpha = 1$, $p = 1$. Hardy and Littlewood [13].

[Using Minkowski's inequality 4.13(4), the proof of (i) is similar to that of Theorem 7.4; (ii) is equivalent to Theorem 7.21 (§ 4.7.6)].

CHAPTER VIII.

Divergence of Fourier series. Gibbs's phenomenon.

8.1. Continuous functions with divergent Fourier series. In Chapter II we proved several conditions ensuring the convergence of Fourier series. Now we will investigate in what degree those tests represent the best possible results. It will appear that, although some improvements are still possible, the problem of the convergence of Fourier series *at individual points* has reached a stage where we can hardly hope for essentially new positive results, if we only use the classical devices of Chapter II. Such tests as Dini's or Dini-Lipschitz's represent a limit beyond which we encounter actual divergence of Fourier series.

The first negative result in the convergence of Fourier series is due to P. Du Bois Reymond (1876) who proved that

There exist continuous functions with Fourier series diverging at a point [1]).

Since that several other examples have been found, and we intend to reproduce two of them. The first is due do Fejér [2]) and is remarkable for its elegance and simplicity. The second method (§ 8.31), propounded by Lebesgue, lies more at the roots of the matter and can be used in many similar problems.

8.11. Fejér's example. It is based on the use of the trigonometrical polynomial

[1]) P. Du Bois Reymond [1].
[2]) Fejér [7].

$$(1)\qquad \begin{aligned} &\frac{\cos \mu x}{n} + \frac{\cos(\mu+1)x}{n-1} + \ldots + \frac{\cos(\mu+n-1)x}{1} - \\ &\qquad - \frac{\cos(\mu+n+1)x}{1} - \ldots - \frac{\cos(\mu+2n)}{n}. \end{aligned}$$

Let us denote it by $Q(x, \mu, n)$, and let $\bar{Q}(x, \mu, n)$ be the conjugate polynomial. Adding up the terms with the same denominator we find that

$$Q(x, \mu, n) = \sin(\mu+n)\, x \sum_{k=1}^{n} \frac{\sin kx}{k},$$

$$\bar{Q}(x, \mu, n) = -\cos(\mu+n)\, x \sum_{k=1}^{n} \frac{\sin kx}{k}.$$

Since the partial sums of the series $\sin x + \frac{1}{2}\sin 2x + \ldots$ are less than a constant C in absolute value (§ 3.23(ii), § 5.11), we have $|Q| \leqslant C$, $|\bar{Q}| \leqslant \bar{C}$, for every x, μ, n. On the other hand, for $x = 0$, the sum of the first n terms of $Q(x, \mu, n)$, which is equal to $1/n + 1/(n-1) + \ldots + 1 > \log n$, is large with n.

Let $\{n_k\}$, $\{\mu_k\}$ be sets of integers which we shall define in a moment, and let $\alpha_k > 0$, $\alpha_1 + \alpha_2 + \ldots < \infty$. The series

$$(2)\qquad \text{a)}\ \sum_{k=1}^{\infty} \alpha_k\, Q(x, \mu_k, n_k), \qquad \text{b)}\ \sum_{k=1}^{\infty} \alpha_k\, \bar{Q}(x, \mu_k, n_k)$$

converge uniformly to continuous sums which we denote by $f(x)$, $g(x)$ respectively. If $\mu_k + 2n_k < \mu_{k+1}$ $(k = 1, 2, \ldots)$, then $Q(x, \mu_k, n_k)$ and $Q(x, \mu_l, n_l)$ do not overlap for $k \neq l$. Similarly $\bar{Q}(x, \mu_k, n_k)$ and $\bar{Q}(x, \mu_l, n_l)$. Therefore, writing every Q and $\bar{Q}$ in (2) in extenso, we represent (2) in the form of trigonometrical series

$$(3)\qquad \text{a)}\ \tfrac{1}{2}a_0 + \sum_{\nu=1}^{\infty}(a_\nu \cos \nu x + b_\nu \sin \nu x), \qquad \text{b)}\ \sum_{\nu=1}^{\infty}(a_\nu \sin \nu x - b_\nu \cos \nu x).$$

Actually the first of them contains only cosines, the second only sines. Denoting the partial sums of these series by $s_n(x)$, $t_n(x)$, we see that $s_{\mu_k-1}(x)$ and $t_{\mu_k-1}(x)$ converge uniformly, so that (3a) is $\mathfrak{S}[f]$ and (3b) is $\mathfrak{S}[g]$. Since $|s_{\mu_k+n_k}(0) - s_{\mu_k-1}(0)| > \alpha_k \log n_k$, the series (3a) will certainly be divergent at $x = 0$ if $\alpha_k \log n_k$ does not tend to 0. Thus

If $\alpha_k = k^{-2}$, $\mu_k = n_k = 2^{k^2}$, the continuous function f defined by (2a) *has a divergent Fourier series.*

It is not difficult to see that both series (3) converge uniformly for $\delta \leqslant |x| \leqslant \pi$, whatever $\delta > 0$. This follows from the fact that the partial sums of $Q(x, \mu_k, n_k)$ and $\overline{Q}(x, \mu_k, n_k)$ are bounded for $0 < \delta \leqslant |x| \leqslant \pi$, uniformly in x, μ_k, n_k (§ 1.22). Since the series (3b), containing only sines, converges for $x = 0$, it converges everywhere.

8.12. *If* $\alpha_k = k^{-2}$, $\mu_k = n_k = 2^{k^3}$, *the continuous function* g *defined by* 8.11(2b) *has a Fourier series which is convergent everywhere, but not uniformly* [1]).

In fact, if $x = \pi/4n$ and $\mu = n$, the sum of the first n terms of $\overline{Q}(x, \mu, n)$ exceeds $(1 + 1/2 + \ldots + 1/n) \sin(\pi/4) > (\log n)/\sqrt{2}$. Therefore $|t_{\mu_k+n_k}(x) - t_{\mu_k-1}(x)| > \alpha_k(\log n_k)/\sqrt{2} \to \infty$ for some x, and this completes the proof. We add a few remarks.

8.13. (i) If we put $\alpha_k = 1/k^2$, $\mu_k = 2^{k^2}$ in 8.11(2), the partial sums $s_n(x)$, $t_n(x)$ are uniformly bounded ($|s_n| < A$, $|t_n| < A$) in $(-\pi, \pi)$, but $\{s_n(0)\}$ oscillates finitely and $\{t_n(x)\}$, which converges everywhere, does not converge uniformly in the neighbourhood of $x = 0$.

(ii) *There exists a power series* $c_0 + c_1 z + \ldots$ *regular for* $|z| < 1$, *continuous for* $|z| \leqslant 1$, *and divergent at* $z = 1$. For $\mathfrak{S}[g] = \overline{\mathfrak{S}}[f]$, and so the power series $c_0 + c_1 z + \ldots$ which reduces to $\mathfrak{S}[f] + i\overline{\mathfrak{S}}[f]$ for $z = e^{ix}$ is an instance in point [2]).

(iii) *There exist continuous functions* $F(x)$ *and* $G(x)$ *such that* $\mathfrak{S}[F]$ *diverges at an everywhere dense set of points, and* $\mathfrak{S}[G]$ *converges everywhere, but in no interval uniformly* [3]).

Let $f(x)$, $g(x)$ be the functions considered in (i), and let $r_1, r_2, r_3, \ldots$ be a set E of points everywhere dense in $(0, 2\pi)$, $\varepsilon_i > 0$, $\varepsilon_1 + \varepsilon_2 + \ldots < \infty$. We put $F(x) = \varepsilon_1 f(x - r_1) + \varepsilon_2 f(x - r_2) + \ldots$, $G(x) = \varepsilon_1 g(x - r_1) + \varepsilon_2 g(x - r_2) + \ldots$, and denote by $F_k(x)$, $G_k(x)$ the k-th partial sums of these series. Let $F(x) = F_k(x) + R_k(x)$, $G(x) = G_k(x) + R_k^*(x)$. The series defining $F(x)$ converges uniformly and we obtain a partial sum of $\mathfrak{S}[F]$ by adding the cor-

---

1) The first example of this singularity is due to Lebesgue.

2) Fejér [7].

3) For the first part of the theorem see P. Du Bois Reymond [1], Fejér [7], for the second Steinhaus [6].

responding partial sums of $\mathfrak{S}[\varepsilon_i f(x-r_i)]$ for $i = 1, 2, \ldots$ Suppose that $\eta > 0$ is given. The partial sums of $\mathfrak{S}[R_k]$ and $\mathfrak{S}[R_k^*]$ are all less than $A(\varepsilon_{k+1} + \varepsilon_{k+2} + \ldots) < \eta$ in absolute value (see (i)), provided that $k = k(\eta)$ is large enough. Since $\mathfrak{S}[F] = \mathfrak{S}[F_k] + \mathfrak{S}[R_k]$, $\mathfrak{S}[G] = \mathfrak{S}[G_k] + \mathfrak{S}[R_k^*]$, we conclude that (1) $\mathfrak{S}[F]$ diverges at any of the points r_i, $1 \leqslant i \leqslant k$, where the oscillation of the partial sums of $\mathfrak{S}[\varepsilon_i f(x-r_i)]$ exceeds η, (2) if $x \,\overline{\in}\, E$, the oscillation of the partial sums of $\mathfrak{S}[F]$ at x is $< \eta$, (3) the oscillation of $\mathfrak{S}[G]$ is less than η at every x. Since η and $1/k$ may be arbitrarily small, we obtain from (1) and (2) that $\mathfrak{S}[F]$ diverges for $x \in E$ and converges for $x \,\overline{\in}\, E$. From (3) we deduce that $\mathfrak{S}[G]$ converges everywhere and it remains only to show that the convergence is non-uniform in the neighbourhood of every r_h. Now, since $\mathfrak{S}[f(x-r_h)]$ converges non-uniformly in the neighbourhood of r_h, so does $\mathfrak{S}[\varepsilon_h g(x-r_h) + R_k^*] = \mathfrak{S}[\varepsilon_h g(x-r_h)] + \mathfrak{S}[R_k^*]$, if $k > h$ is large enough. We have $G = [G_k - \varepsilon_h g(x-r_h)] + [\varepsilon_h g(x-r_h) + R_k^*]$ and, since $\mathfrak{S}[G_k - \varepsilon_h g(x-r_h)]$ converges uniformly in a neighbourhood of r_h, the convergence of $\mathfrak{S}[G]$ cannot be uniform there, and this completes the proof.

8.14. In the preceding section we proved more than we set out to prove since we showed that, for any enumerable set E, there exists a continuous f, such that $\mathfrak{S}[f]$ diverges in E and converges outside E [1]). The problem of existence of a continuous f with $\mathfrak{S}[f]$ divergent everywhere, or almost everywhere, is not solved yet and seems to be exceedingly difficult. However it is a very simple matter to construct a continuous f with $\mathfrak{S}[f]$ divergent in a non-enumerable set of points. Let $r_1, r_2, \ldots$ be now the sequence containing any rational point of the interval $(0, 2\pi)$ infinitely many times and let $f(x) = \sum_{k=1}^{\infty} k^{-2} Q(x - r_k, 2^{k^3}, 2^{k^3})$. Here f is continuous, and to obtain $\mathfrak{S}[f]$ we simply replace every Q by the expression 8.11(1). At any rational point, $\mathfrak{S}[f]$ will contain infinitely many blocks of terms with sums exceeding $k^{-2} \log 2^{k^3}$ for some, arbitrarily large, values of k. It follows that $\mathfrak{S}[f]$ has the partial sums unbounded at an everywhere dense set of points. We know that the set of points at which a sequence of continuous functions $s_n(x)$ is bounded is a sum $F_1 + F_2 + \ldots$ of closed

---

1) Steinhaus [7]. See also Neder [1], Zalcwasser [1].

sets (§ 6.11). In our case no F_i contains an interval, and the sum $F_1 + F_2 + \dots$ of non-dense sets is of the first category. It is known that the sets complementary to sets of the first category contain perfect subsets, and therefore are of the power of the continuum.

8.2. A theorem of Faber and Lebesgue. We shall show that the Dini-Lipschitz condition cannot be generalized. *There exist two continuous functions $f(x)$, $g(x)$, both having the modulus of continuity $O(1/\log 1/\delta)$ and such that $\mathfrak{S}[f]$ diverges for $x=0$, $\mathfrak{S}[g]$ converges everywhere but not uniformly* [1]). We define f and g as the sums of the series 8.11(2) respectively, with $\alpha_k = 2^{-k}$, $\mu_k = n_k = 2^{2^k}$. The argument used in § 8.11 shows that $\mathfrak{S}[f]$ oscillates finitely at $x=0$, and that $\mathfrak{S}[g]$ converges non-uniformly in the neighbourhood of $x=0$. To prove the inequalities for $\omega(\delta; f)$ and $\omega(\delta; g)$, e. g. for the former, let $\nu = \nu(h)$ be the largest integer k such that $2^{2^k} \leqslant 1/h$, where $h > 0$. Break up the sum defining f into two parts $f_1(x)$, $f_2(x)$, the latter consisting of terms with indices $> \nu$. We have then $|f_2(x+h) - f_2(x)| \leqslant 2C\,(2^{-\nu-1} + 2^{-\nu-2} + \dots) = 4C.\,2^{-\nu-1} \leqslant 4C/\log 1/h$. A simple calculation shows that

$$Q'(x, \mu, n) = -(\mu + n)\,\overline{Q}(x, \mu, n) - \sin \mu x - \dots - \sin(\mu + n - 1)\,x + \\ + \sin(\mu + n + 1)\,x + \dots + \sin(\mu + 2n)\,x,$$

so that $|Q'(x, \mu, n)| \leqslant (\mu + n)\,C + 2n < (\mu + n)(C + 2) = nC'$ if we suppose that $\mu = n$, $2(C+2) = C'$. By the mean-value theorem we see that $|f_1(x+h) - f_1(x)|$ does not exceed

$$C'h\,[2^{-1}\,2^{2^1} + 2^{-2}\,2^{2^2} + \dots + 2^{-\nu}\,2^{2^\nu}] = O(h\,2^{-\nu}\,2^{2^\nu})\ ^{2)} =$$

$$= O\,(2^{-\nu}) = O\,(1/\log 1/h).$$

Therefore $|f(x+h) - f(x)| \leqslant |f_1(x+h) - f_1(x)| + |f_2(x+h) - f_2(x)| = O(1/\log 1/h)$ and the theorem is established. Arguing as in § 8.13(iii), we can make $\mathfrak{S}[f]$ diverge in a set everywhere dense, and $\mathfrak{S}[g]$ converge non-uniformly in every interval.

---

[1]) Faber [1], Lebesgue [1].

[2]) We use here the following proposition: if, for a positive sequence $\{m_k\}$, we have $m_{k+1}/m_k > q > 1$, then $m_1 + m_2 + \dots + m_k = O(m_k)$; for $m_1 + m_2 + \dots + m_{k-1} + m_k < m_k(1 + q^{-1} + q^{-2} + \dots + q^{-k}) < m_k/(1 - q^{-1})$.

8.3. Lebesgue's constants. This name is given to the numbers

$$(1)\qquad L_n=\frac{1}{\pi}\int_{-\pi}^{\pi}|D_n(t)|\,dt=\frac{2}{\pi}\int_0^{\pi}\frac{|\sin(n+\frac{1}{2})t|}{2\sin\frac{1}{2}t}\,dt.$$

Let $s_n(x;f)$ denote the n-th partial sum of $\mathfrak{S}[f]$. It is plain that, if $|f|\leqslant 1$, then $|s_n(x;f)|\leqslant L_n$, and for the function $f(t)=\operatorname{sign} D_n(t)$ we actually have $s_n(0;f)=L_n$. The latter function is discontinuous at a finite number of points, but, smoothing this function slightly at the points of discontinuity, we can obtain a continuous f, $|f|\leqslant 1$, such that $s_n(0;f)>L_n-\varepsilon$, whatever $\varepsilon>0$. Thus, for a fixed n, L_n is the upper bound of $|s_n(x;f)|$ for all x and continuous f, $|f|\leqslant 1$. For this reason it is interesting to investigate the behaviour of L_n as $n\to\infty$. We will prove that $L_n\simeq(4/\pi^2)\log n$ as $n\to\infty$ [1]).

Since the function $\frac{1}{2}\operatorname{ctg}\frac{1}{2}t-1/t$ is bounded for $|t|\leqslant\pi$, and $|\sin nt|\leqslant|nt|$, we have

$$L_n=\frac{2}{\pi}\int_0^{\pi}\frac{|\sin nt|}{2\operatorname{tg}\frac{1}{2}t}\,dt+O(1)=\frac{2}{\pi}\int_0^{\pi}\frac{|\sin nt|}{t}\,dt+O(1)=$$

$$=\frac{2}{\pi}\sum_{k=1}^{n-1}\int_{k\pi/n}^{(k+1)\pi/n}\frac{|\sin nt|}{t}\,dt+O(1)=\frac{2}{\pi}\int_0^{\pi/n}\sin nt\left\{\sum_{k=1}^{n-1}\frac{1}{t+k\pi/n}\right\}dt+O(1).$$

The sum in curly brackets, contained between the numbers $n\pi^{-1}(1+1/2+\ldots+1/(n-1))$ and $n\pi^{-1}(1/2+1/3+\ldots+1/n)$, is equal to $\pi^{-1}n[\log n+O(1)]$ (§ 1.74). Since the integral of $\sin nt$ over $(0,\pi/n)$ is equal to $2/n$, we have $L_n=(4/\pi^2)\log n+O(1)\simeq(4/\pi^2)\log n$.

8.31. We have proved that, if n is large enough, there exists a continuous $f(x)=f_n(x)$, $|f_n|\leqslant 1$, such that $s_n(0;f)$ is large. This function depends on n. To obtain a fixed f with $s_n(0;f)$ unbounded we appeal to Theorem 4.56(iv). If we replace in it $y_n(t)$ by $D_n(t)$, $x(t)$ by $f(t)$, and use the fact that $L_n\to\infty$, we deduce that there is a continuous function $f(x)$ with $\overline{\lim}\,|s_n(0;f)|=+\infty$, i. e. Theorem 8.1 [2]).

---

1) Fejér [8].

2) Theorem 4.56(iv) (which is due to Lebesgue [2]) lies rather deep, and in the case $y_n(t)=D_n(t)$ it is not difficult to prove it directly. We refer the reader to Lebesgue's *Leçons*.

8.32. Let λ_n be any sequence tending to $+\infty$ more slowly than $\log n$. Since the integral of $|D_n(t)|/\lambda_n$ over $(-\pi, \pi)$ tends to $+\infty$, applying Theorem 4.56(iv) again we have:

For any sequence $\lambda_n = o(\log n)$ there exists a continuous f such that $s_n(0; f) > \lambda_n$ for infinitely many n. In § 2.73 we proved that, for any continuous f, $s_n(x; f) = o(\log n)$, uniformly in x. Now we see that this result cannot be improved.

The above theorem can also be established by the method of § 8.11.

8.33. Applying Theorem 4.55 *in its most general form* to the proof of Theorem 4.56(iv), we obtain a result from which we conclude that the set of continuous functions f with $\mathfrak{S}[f]$ convergent at the point 0, or at any fixed point, forms a set of the first category in the space C of all continuous and periodic functions. Thus the set of continuous functions with Fourier series convergent at some rational point or another is again of the first category. In other words, *if we reject from the space C a set of the first category, the Fourier series of the remaining functions have points of divergence everywhere dense.*

8.34. As a last application of Theorem 4.56(iv) we shall show that, in a sense, the Dini condition of § 2.4 cannot be improved: *Given any continuous $\mu(t) \geqslant 0$, such that $\mu(t)/t$ is not integrable in the neighbourhood of $t = 0$, we can find a continuous function f, such that $|f(t) - f(0)| \leqslant \mu(t)$ for small $|t|$, and none the less $\mathfrak{S}[f]$ diverges at $t = 0$.*

Let $s_n^*(x; f)$ be the modified partial sums of $\mathfrak{S}[f]$ (§ 2.3). Put $\chi_n(t) = \mu(t) \sin nt/2 \operatorname{tg} \frac{1}{2} t$. If $\mathfrak{M}[\chi_n] \neq O(1)$, we can find a continuous $g(x)$, $|g| \leqslant 1$, such that the integral of $\chi_n(t)\, g(t)$ over $(-\pi, \pi)$ is unbounded as $n \to \infty$. This means that $\mathfrak{S}[f]$, where $f(x) = g(x)\, \mu(x)$, diverges at the point 0. Since we may freely suppose that $\mu(0) = 0$, we have $|f(t) - f(0)| = |f(t)| \leqslant \mu(t)$.

To justify our assumption that $\mathfrak{M}[\chi_n] \neq O(1)$, we prove the following lemma: *If $\alpha(x)$ is bounded, $\beta(x)$ integrable, both periodic, then*

$$(1) \qquad I_n = \int_{-\pi}^{\pi} \alpha(nx)\, \beta(x)\, dx \to \int_{-\pi}^{\pi} \alpha(x)\, dx \int_{-\pi}^{\pi} \beta(x)\, dx$$

as $n \to \infty$ [1]). We begin by the following observation, the proof of which may be left to the reader: If, for every $\varepsilon > 0$, we have $\beta = \beta_1 + \beta_2$, where $\mathfrak{M}[\beta_1] < \varepsilon$ and the relation (1) holds for β_2 and any bounded α, then (1) is true. Now (1) is certainly true if β is the characteristic function of a set E consisting of a finite number of intervals. Therefore it holds true when E is open, or, more generally, measurable. Thence we pass to the case of β assuming only a finite number of values. Since we can approximate uniformly to any bounded β by such functions, we conclude the truth of (1) for β bounded. If β is integrable, we put $\beta = \beta_1 + \beta_2$, where β_2 is bounded and $\mathfrak{M}[\beta_1]$ small.

Let us now put $\alpha(t) = |\sin t|$, $\beta(t) = \mu(t)/|2 \operatorname{tg} \frac{1}{2} t|$ for $0 < \varepsilon \leqslant |t| \leqslant \pi$, $\beta(t) = 0$ elsewhere, and denote the corresponding integrals I_n by $I_n(\varepsilon)$. Since $\mathfrak{M}[\chi_n] \geqslant I_n(\varepsilon)$, we have the inequalities $\underline{\lim}\, \mathfrak{M}[\chi_n] \geqslant \underline{\lim}\, I_n(\varepsilon) = \lim I_n(\varepsilon)$. The function $\mu(t)/2 \operatorname{tg} \frac{1}{2} t$ being non-integrable, we may make $\lim I_n(\varepsilon)$ as large as we please, if only ε is small enough. This shows that $\mathfrak{M}[\chi_n] \to \infty$, and the theorem is established.

The case $\mu(t) = o(\log 1/|t|)^{-1}$ (we may put, for example, $\mu(t) = (\log 1/|t| \log\log 1/|t|)^{-1}$ for small $|t|$) is of special interest in view of the Dini-Lipschitz test (§ 2.71).

Consider a continuous function $f(t)$ with $\mathfrak{S}[f]$ divergent at the point 0, and such that $f(0) = 0$, $f(t) = o(\log 1/|t|)^{-1}$. Let $f_1(t) = f(t)$, $f_2(t) = 0$ for $0 \leqslant t \leqslant \pi$, and $f_1(t) = 0$, $f_2(t) = f(t)$ for $-\pi \leqslant t \leqslant 0$. Since $f = f_1 + f_2$, it follows that at least one of the functions f_1, f_2, say f_1, has a Fourier series divergent at the point 0. Consider the interval $(a, b) = (-\pi/4, 0)$. It is plain that the modulus of continuity of the function f_1 in (a, b) is $o(\log 1/\delta)^{-1}$ and that $f(a-t) - f(a) = o(\log 1/t)^{-1}$, $f(b+t) - f(b) = o(\log 1/t)^{-1}$ as $t \to +0$. In spite of that, $\mathfrak{S}[f_1]$ does not converge uniformly in the interval (a, b). This result justifies the last remark of § 2.72.

8.35. Lebesgue's constants may be defined for any method of summation if we replace $D_n(t)$ in 8.3(1) by the corresponding kernel. In the case of the method $(C, 1)$, or Abel's method, Lebesgue's constants are all equal to 1. As regards the constants

[1]) Fejér [8]. This lemma will be applied only in a special case of α continuous, and β continuous except at a finite number of points. We prove it in the general case since it embraces Theorem 2.211.

$L_n^{(k)}$ corresponding to the metod (C, k), $0 < k < 1$, the following result has been proved. *$L_n^{(k)}$ tends to a finite number $L^{(k)} > 1$, as $n \to \infty$. For any $0 < k < 1$, there exists an $f(x)$, $|f| \leqslant 1$, such that* $\overline{\lim} \, |\sigma_n^k(0; f)| = L^{(k)}$ [1]).

8.4. Kolmogoroff's example. *There exists an integrable function $f(x)$ such that $\mathfrak{S}[f]$ diverges everywhere* [2]).

Let $f_1(x)$, $f_2(x)$, ... be a sequence of trigonometrical polynomials of orders $\nu_1 < \nu_2 < \ldots$, with the following properties (i) $f_n(x) \geqslant 0$, (ii) $\int_0^{2\pi} f_n(x)\,dx = 2\pi$. Suppose, moreover, that to every f_n corresponds an integer λ_n, where $0 < \lambda_n \leqslant \nu_n$, a number $A_n > 0$, and a point set E_n, such that (iii) if $x \in E_n$, there is an integer $k = k_x$, $\lambda_n \leqslant k \leqslant \nu_n$, for which $s_k(x; f_n) > A_n$, (iv) $A_n \to \infty$, (v) $\lambda_n \to \infty$, (vi) $E_1 \subset E_2 \subset \ldots$, $E_1 + E_2 + \ldots = (0, 2\pi)$. *Under these conditions, if $\{n_k\}$ tends to ∞ sufficiently rapidly, the Fourier series of the function*

$$f(x) = \sum_{k=1}^{\infty} f_{n_k}(x) / \sqrt{A_{n_k}} \tag{1}$$

diverges everywhere.

First of all the series in (1) converges almost everywhere to an integrable sum provided that the series $1/\sqrt{A_{n_1}} + 1/\sqrt{A_{n_2}} + \ldots$ converges. This follows from the fact that series with non-negative terms can be integrated term by term. Let us put $n_1 = 1$ and assume that the numbers $n_1, n_2, \ldots, n_{i-1}$ have already been defined. The number n_i will be defined as the least integer satisfying the conditions:

$$\text{(a)}\quad \lambda_{n_i} > \nu_{n_{i-1}}, \qquad \text{(b)}\quad A_{n_i} > 4A_{n_{i-1}}, \qquad \text{(c)}\quad \sqrt{A_{n_i}} > \nu_{n_{i-1}}.$$

From (b) we deduce the convergence of $1/\sqrt{A_{n_1}} + 1/\sqrt{A_{n_2}} + \ldots$, so that $f(x)$ exists and is integrable. To prove the divergence of $\mathfrak{S}[f]$, let x be an arbitrary point of E_{n_i} and let $f = u + v + w$, where u is the $(i-1)$-st partial sum of the series (1), and $v = f_{n_i}/\sqrt{A_{n_i}}$;

---

[1]) Cramér [1].

[2]) Kolmogoroff [6]. The construction of the text is slightly different from that of the original paper. The modifications have been suggested to me by Mr. Kolmogoroff.

hence $s_k(x; f) = s_k(x; u) + s_k(x; v) + s_k(x; w)$. In virtue of (iii) there is a $k = k_x$, $\lambda_{n_l} \leqslant k \leqslant \nu_{n_l}$, such that

$$(2) \qquad s_k(x; v) \geqslant \sqrt{A_{n_l}}.$$

From (a) and (i) we see that

$$(3) \qquad s_k(x; u) = u(x) \geqslant 0.$$

Finally, since for any integrable g we have $|s_k(x; g)| \leqslant (2k+1)\, \mathfrak{M}[g; 0, 2\pi]/\pi$, we find that $|s_k(x; w)| \leqslant 2(2k+1)\,(1/\sqrt{A_{n_{l+1}}} + 1/\sqrt{A_{n_{l+2}}} + \dots) < 12k/\sqrt{A_{n_{l+1}}} \leqslant 12\nu_{n_l}/\sqrt{A_{n_{l+1}}} \leqslant 12$. From this and the inequalities (2), (3), we conclude that $s_k(x; f) \geqslant \sqrt{A_{n_l}} - 12$. Since every $x \in (0, 2\pi)$ belongs to E_{n_l} for all l sufficiently large, the result follows.

8.401. It remains to construct the polynomials f_n and to show that they possess the required properties; this is the most fundamental part of the proof. The function $f_n(x)$ will be defined as a sum of two polynomials $\varphi(x)$ and $\psi(x)$.

Let us fix n, put $x_i = 2\pi i/(2n+1)$, $i = 0, 1, \dots, 2n$, and consider the intervals $I_i = (x_i - \delta, x_i + \delta)$. If δ is small enough, there is a non-negative trigonometrical polynomial $\varphi(x)$ of order $M > n$, with constant term equal to $\frac{1}{2}$, and such that $\varphi(x) \geqslant n$, say, in the intervals I_i. For it is sufficient to put $\varphi(x) = K_m\{(2n+1)x\}$, where K_m denotes Fejér's kernel and m is large enough. Since we may take δ as small as we please, we may suppose that $D_M(x) \geqslant 0$ in the interval $(-\delta, \delta)$, where D_M denotes Dirichlet's kernel.

Next we put

$$\psi(x) = \frac{1}{n+1} \sum_{i=0}^{n} K_{m_i}(x - x_{2i}),$$

where $M \leqslant m_0 < m_1 < \dots$; the numbers $m_0, m_1, \dots$ will be defined later. If $m_j \leqslant k < m_{j+1}$, then

$$s_k(x; \psi) = \frac{1}{n+1} \sum_{i=0}^{j} K_{m_i}(x - x_{2i}) +$$

$$+ \frac{1}{n+1} \sum_{i=j+1}^{n} \left\{ \frac{1}{2} + \sum_{l=1}^{k} \frac{m_i - l + 1}{m_i + 1} \cos l\,(x - x_{2i}) \right\}$$

Since $m_i - l + 1 = (m_i - k) + (k - l + 1)$ and $K_k(x) \geqslant 0$, we obtain

$$(1)\quad s_k(x;\psi) \geqslant \frac{1}{n+1}\sum_{i=j+1}^{n}\frac{m_i-k}{m_i+1}D_k(x-x_{2i}), \qquad m_j \leqslant k < m_{j+1}.$$

8.402. Let us denote the intervals $(x_i+\delta, x_{i+1}-\delta)$ by I'_i, $i = 0, 1, 2, \dots, 2n$, and suppose that $x \in I'_{2j}$ or that $x \in I'_{2j+1}$; in particular $x_{2i} < x < x_{2i+2}$. If $2k+1$ is a multiple of $2n+1$, then $\sin(k+\frac{1}{2})(x-x_{2i})$ has the same value for every i, and from 8.401(1) we obtain

$$(1)\quad s_k(x;\psi) \geqslant \frac{\sin(k+\frac{1}{2})(x_{2j+2}-x)}{n+1}\sum_{i=j+1}^{n}\frac{m_i-k}{m_i+1}\cdot\frac{1}{2\sin\frac{1}{2}(x_{2i}-x)}.$$

It is not difficult to prove that, if the numbers $m_0, m_1, \dots$ increase sufficiently rapidly, then, to every x belonging either to I'_{2j} or to I'_{2j+1}, corresponds an integer $k = k_x$ satisfying the inequalities $m_j \leqslant k < \frac{1}{2}m_{j+1}$, $\sin(k+\frac{1}{2})(x_{2j+2}-x) \geqslant \frac{1}{2}$, and such that $2k+1$ is a multiple of $2n+1$. Let us take this result for granted; we shall return to it later. Taking such a value for k, we obtain from (1)

$$s_k(x;\psi) \geqslant \frac{1}{2}\cdot\frac{1}{n+1}\sum_{i=j+1}^{n}\frac{1}{2(x_{2i}-x)} > \frac{1}{2n+2}\sum_{i=j+1}^{n}\frac{2n+1}{8\pi(i-j)},$$

i. e. $s_k(x;\psi) \geqslant C_1\log(n-j)$, $C, C_1, C_2, \dots$ denoting positive absolute constants. If $j \leqslant n-\sqrt{n}$, then $s_k(x;\psi) \geqslant \frac{1}{2}C_1\log n = C\log n$.

8.403. Let us put $f_n(x) = \varphi(x) + \psi(x)$. If $x \in I'_{2j}$, or $x \in I'_{2j+1}$, $j \leqslant n-\sqrt{n}$, there is an integer $k \geqslant m_j \geqslant m_0 \geqslant M$ such that $s_k(x;\psi) \geqslant C\log n$. Hence we have $s_k(x;f_n) = s_k(x;\varphi) + s_k(x;\psi) = \varphi(x) + s_k(x;\psi) \geqslant C\log n$.

Now we shall investigate the behaviour of $s_k(x;f_n)$ in the intervals I_l. We shall show that $s_M(x;f_n) \geqslant \frac{1}{2}n$ for $x \in I_l$ and n sufficiently large. The right-hand side of the equation $s_M(x;f_n) = s_M(x;\varphi) + s_M(x;\psi)$ consists of two terms, the first of which exceeds n for $x \in I_l$, and we will show that, if $x \in I_l$, the second term is dominated by the first (this is just the reason why we define f_n as $\varphi + \psi$). More precisely, we shall prove the inequality $s_M(x;\psi) > -C_2\log n$ for $x \in I_l$ and $n > 1$, so that $s_M(x;f_n) \geqslant n - C_2\log n > \frac{1}{2}n$ for $x \in I_l$, $n > n_0$.

We first suppose that l is even, $l = 2h$. If $k = M = m_0$, we have the formula 8.401(1) with $j = -1$. If $x \in I_{2h}$, the term $i = h$ in the sum on the right is positive in virtue of the condition imposed on the intervals I_i. If this term is omitted, the inequality 8.401(1) holds à fortiori. Since $|D_k(u)| \leqslant \pi/|u|$, for $|u| < \pi$, and since $|x - x_{2i}| > 2\pi|h-i|/(2n+1)$, we obtain that

$$(1)\quad s_M(x;\psi) \geqslant -\frac{1}{n+1}\sum_{i=0}^{n}{}' \frac{2n+1}{2|h-i|} > -C_3 \log n, \quad n > 1, \quad x \in I_{2h},$$

where $'$ denotes that $i \neq h$.

If l is odd, $l = 2h+1$, we again have the inequality 8.401(1) with $j = -1$, $k = M$. It is not difficult to see that $|x - x_{2i}|$ exceeds a constant multiple of $|h-i|/(2n+1)$, and, arguing as in the previous case, we obtain that $s_M(x;\psi) > -C_4 \log n$, for $x \in I_{2h+1}$, $n > 1$. This, together with (1), gives $s_M(x;\psi) > -C_2 \log n$, where $x \in I_l$, $n > 1$, $C_2 = \text{Max}\,(C_3, C_4)$. Hence, as we have already observed, $s_M(x; f_n) > \frac{1}{2} n$ if $x \in I_l$, $n > n_0$.

Collecting the results and observing that $C \log n < \frac{1}{2} n$ for n sufficiently large, we obtain that *to every x in the interval* (E_n) $0 \leqslant x \leqslant 4\pi(n - \sqrt{n})/(2n+1)$ *corresponds an integer $k > n$, such that $s_k(x; f_n) > C \log n$, $n > n_1$.* The reader will have no difficulty in verifying that the functions f_n satisfy the conditions of the lemma established in § 8.4, at least for n sufficiently large.

8.404. There is one point in the preceding argument which requires explanation. We must show that, if the numbers $m_0, m_1, \dots$ increase sufficiently rapidly, then, to every x belonging to $I'_{2j} + I'_{2j+1}$, corresponds an integer k satisfying the inequalities $m_j \leqslant k < \frac{1}{2} m_{j+1}$, $\sin(k+\frac{1}{2})(x_{2j+2} - x) \geqslant \frac{1}{2}$, and such that $2k+1$ is divisible by $2n+1$. Let us put $2k+1 = \rho(2n+1)$, so that ρ is odd, and $x_{2j+2} - x = 4\pi\theta/(2n+1)$. Then $\sin(k+\frac{1}{2})(x_{2j+2}-x) = \sin 2\pi\rho\theta$, and x belongs to $I'_{2j} + I'_{2j+1}$ if and only if θ belongs to the sum of intervals $\eta \leqslant \theta \leqslant \frac{1}{2} - \eta$, $\frac{1}{2} + \eta \leqslant \theta \leqslant 1 - \eta$, where η is positive and depends on δ and n.

Let $m_0 = M$, and suppose that $m_0, m_1, \dots, m_j$ have already been defined. It is sufficient to show that, if ρ_0 is a fixed odd integer, then there is a number ν such that, if θ belongs to $(\eta, \frac{1}{2} - \eta) + (\frac{1}{2} + \eta, 1 - \eta)$, we have $\sin 2\pi\rho\theta \geqslant \frac{1}{2}$ for an odd integer ρ satisfying the inequality $\rho_0 \leqslant \rho \leqslant \rho + \nu$. For, if m'_j denotes a number such that for every $x \in I'_{2j} + I'_{2j+1}$ there is an integer k, $m_j \leqslant k \leqslant m'_j$ such that $2k+1$ is a multiple of $(2n+1)$, and that $\sin(k+\frac{1}{2})(x_{2j+2} - x) \geqslant \frac{1}{2}$, then we may take for m_{j+1} any integer greater than $2m'_j$.

Now consider the points $\rho\theta$ where ρ runs through the sequence $\rho_0, \rho_0+2, \rho_0+4, \ldots$ If ρ is increased by 2, $\theta\rho$ increases by 2θ, i. e. by a number the fractional part of which belongs to the interval $(2\eta, 1-2\eta)$. Consider the following three cases (i) $2\theta \in (2\eta, {}^1/_3)$, (ii) $2\theta \in ({}^1/_3, {}^2/_3)$, (iii) $2\theta \in ({}^2/_3, 1-2\eta)$. In case (i) the situation is fairly simple; for the length of the interval $({}^1/_{12}, {}^5/_{12})$ is equal to ${}^1/_3$, and so after a bounded number of steps the point $\rho\theta$ will certainly fall into this interval i. e. we shall have $\sin 2\pi\rho\theta \geqslant \frac{1}{2}$. In case (iii) the argument is similar.

In case (ii) the situation is slightly less simple for, if $\theta\rho_0$ and 2θ are both very near (mod 1) to the number ${}^1/_2$, the sequence $\theta\rho$, $\rho \geqslant \rho_0$, may stay outside the interval $({}^1/_{12}, {}^5/_{12})$ for a long time. Consider the cases (ii') $2\theta \in ({}^1/_3, {}^5/_{12})$, (ii'') $2\theta \in ({}^5/_{12}, {}^7/_{12})$, (ii''') $2\theta \in ({}^7/_{12}, {}^2/_3)$. In cases (ii') and (ii'''), 4θ belongs to the intervals $({}^2/_3, {}^5/_6)$, $({}^1/_6, {}^1/_3)$ respectively, and so, arguing as before, we see that, after an e v e n number of steps, $\theta\rho$ will fall into the interval $({}^1/_{12}, {}^5/_{12})$.

Now suppose that $2\theta \in ({}^5/_{12}, {}^7/_{12}) \subset ({}^1/_3, {}^2/_3)$, i. e. θ belongs either to $({}^2/_{12}, {}^4/_{12})$ or to $({}^8/_{12}, {}^{10}/_{12})$, e. g. to the former interval. It is easy to see that, if m is even and positive, and if $m\theta$ belongs either to $({}^1/_{12}, {}^5/_{12})$ or to $({}^7/_{12}, {}^{11}/_{12})$, then, after a bounded number of steps, the point $\rho\theta$, $\rho \geqslant \rho_0$, will reach the interval $({}^1/_{12}, {}^5/_{12})$. Now we observe that the numbers $\rho_0 - 1$, $\rho_0 + 1$, $2\rho_0$ are even and that (a) if $\rho_0\theta \in ({}^1/_{12}, {}^5/_{12})$, we may put $\rho = \rho_0$, (b) if $\rho_0\theta \in ({}^5/_{12}, {}^7/_{12})$, then $(\rho_0 - 1)\,\theta \in ({}^1/_{12}, {}^5/_{12})$, (c) if $\rho_0\theta \in ({}^7/_{12}, {}^8/_{12})$, then $2\rho_0\theta \in ({}^2/_{12}, {}^4/_{12})$, (d) if $\rho_0\theta \in ({}^8/_{12}, {}^{10}/_{12})$, then $(\rho_0+2)\,\theta \in ({}^1/_{12}, {}^5/_{12})$, (e) if $\rho_0\theta \in ({}^{10}/_{12}, {}^{11}/_{12})$, then $2\rho_0\,\theta \in ({}^8/_{12}, {}^{10}/_{12})$, (f) if $\rho_0\theta \in ({}^{11}/_{12}, 1) + (0, {}^1/_{12})$, then $(\rho_0+1)\,\theta \in ({}^1/_{12}, {}^5/_{12})$.

The case $2\theta \in ({}^5/_{12}, {}^7/_{12})$, $\theta \in ({}^8/_{12}, {}^{10}/_{12})$ may be dealt with in the same way and Theorem 8.4 is established completely.

8.5. Gibbs's phenomenon. We shall now investigate the behaviour of the partial sums $d_n(x)$ of the series

$$(1) \qquad \sum_{\nu=1}^{\infty} \frac{\sin \nu x}{\nu} = d(x) = \tfrac{1}{2}(\pi - x) \qquad (0 < x < 2\pi)$$

in the neighbourhood of $x = 0$. Suppose, as we may, that $x > 0$. Since $\frac{1}{2}\operatorname{ctg}\frac{1}{2}t - 1/t$ is of bounded variation over $(0, \pi)$, we have

$$\tfrac{1}{2}x + d_n(x) = \int_0^x D_n(t)\,dt = \int_0^x \frac{\sin nt}{2\operatorname{tg}\frac{1}{2}t}\,dt + o(1) =$$

$$= \int_0^x \frac{\sin nt}{t}\,dt + o(1) = \int_0^{nx} \frac{\sin t}{t}\,dt + o(1),$$

where the last term tends to 0 uniformly in x (§ 2.213). From this we deduce the approximate formula

$$(2) \qquad d_n(x) = \int_0^{nx} \frac{\sin t}{t}\, dt + o\,(1),$$

where the error is $< \varepsilon$, provided that $x < \varepsilon$, $n > n_0(\varepsilon)$. Let us put $\varphi\,(u) = \int_0^u \frac{\sin t}{t}\, dt$. The integrals of $(\sin t)/t$ over the intervals $(k\pi, (k+1)\,\pi)$ decrease in absolute value and are of alternating sign when k runs through the values $0, 1, 2, \dots$ This shows that the curve $y = \varphi\,(x)$ has a wave-like shape with maxima $M_1 > M_3 > M_5 > \dots$ attained at the points $\pi, 3\pi, 5\pi, \dots$ and minima $m_2 < m_4 < m_6 < \dots$ at $2\pi, 4\pi, \dots$ From the relation $d_n(x) \to \frac{1}{2}\,(\pi - x)$, and the equation $d_n(x) = -\frac{1}{2}\,x + \varphi\,(nx) + o\,(1)$, we see that $\varphi\,(u) \to \frac{1}{2}\pi$ as $u \to \infty$, i. e.

$$\int_0^\infty \frac{\sin t}{t}\, dt = \frac{\pi}{2}.$$

Substituting $x = \pi/n$ in the formula (2), we obtain that $d_n(\pi/n) \to \varphi\,(\pi) > \varphi\,(\infty) = \frac{1}{2}\pi$. Thus, although $d_n(x)$ tends to $d\,(x) \leqslant \frac{1}{2}\pi$ for every fixed x, $0 < x < \pi$, the curves $y = d_n(x)$, which pass through the point $(0, 0)$, condense to the interval $0 \leqslant y \leqslant \varphi\,(\pi)$ on the y-axis, transcending the interval $0 \leqslant y \leqslant d\,(+0)$ in the ratio

$$\frac{2}{\pi} \int_0^\pi \frac{\sin t}{t}\, dt = 1.089490\dots$$

Since the $d_n(x)$ are odd functions of x, a similar situation occurs in the left-hand neighbourhood of $x = 0$, where the curves $y = d_n(x)$ condense to the interval $-\varphi\,(\pi) \leqslant y \leqslant 0$ [1]). This phenomenon is called Gibbs's phenomenon and may be described, quite generally, as follows. Let a sequence $\{f_n(x)\}$ converge to a function $f\,(x)$ for $x_0 < x \leqslant x_0 + h$, say. If, for n and $1/(x - x_0)$ tending to $+\infty$ independently of each other, $\overline{\lim}\, f_n(x) > f\,(x_0 + 0)$, or if $\underline{\lim}\, f_n(x) < f\,(x_0 + 0)$, we say that $\{f_n(x)\}$ presents Gibbs's phenomenon in the right-hand neighbourhood of the point x_0. A similar definition holds for the left-hand neighbourhood [2]).

---

[1]) For interesting graphs and a more detailed discussion we refer the reader to Carslaw's, *Introduction to the Theory of Fourier Series and Integrals.*

[2]) See Zalcwasser [1], where a discussion of some problems connected with Gibbs's phenomenon is given.

8.51. Let $f(x)$ be an arbitrary function having a simple discontinuity at a point ξ: $f(\xi+0)-f(\xi-0)=l\neq 0$. The function $\Delta(x)=f(x)-l\cdot d(x-\xi)/\pi$ is continuous at ξ. Suppose that $\mathfrak{S}[\Delta]$ converges uniformly at the point ξ (§ 2.601). The behaviour of $s_n(x;f)$ in the neighbourhood of ξ will then, in a sense, be dominated by the behaviour of $s_n(x;l\cdot d(x-\xi)/\pi)$, and so Gibbs's phenomenon will occur. Thus, in particular, *if f is of bounded variation, $\mathfrak{S}[f]$ will present Gibbs's phenomenon at every point of simple discontinuity of f* [1].

8.52. The formula 8.5(2) has interesting applications [2]. Suppose that $f(x)$ is of bounded variation and ξ a point of discontinuity of f. Let $\{h_m\}$ be a sequence of numbers such that $mh_m\to H$. Making the decomposition $f(x)=\Delta(x)+l\cdot d(x-\xi)/\pi$, we find the formula

$$s_n(\xi+h_n)\to\frac{f(\xi+0)+f(\xi-0)}{2}+\frac{f(\xi+0)-f(\xi-0)}{2}\cdot\frac{2}{\pi}\int_0^H\frac{\sin t}{t}\,dt,$$

where $s_n(x)=s_n(x;f)$. Taking for H one of the infinitely many roots of the equation $\varphi(u)=\pi/2$ (in particular $H=\infty$), we obtain the formulae: $s_n(\xi+h_n)\to f(\xi+0)$, $s_n(\xi-h_n)\to f(\xi-0)$, where $h_n=H/n$ if H is finite and, for example, $h_n=1/\sqrt{n}$ if $H=\infty$. From these formulae we obtain, in particular, the value of the jump $f(\xi+0)-f(\xi-0)$.

8.6. Theorems of Rogosinski [3]. In the preceding paragraph we obtained certain results concerning the behaviour of $s_n(\xi+h_n;f)$, provided that f was of bounded variation. It will appear that similar results hold in the general case if we consider the symmetric expressions $\frac{1}{2}[s_n(\xi+h_n)+s_n(\xi-h_n)]$ instead of $s_n(\xi+h_n)$.

8.61. (i) *If $\alpha_n=O(1/n)$ and if the series* 8.11(3a) [4] *converges at a point ξ, to s, then $\frac{1}{2}[s_n(\xi+\alpha_n)+s_n(\xi-\alpha_n)]\to s$* (ii) *If this series is summable* $(C,1)$ *at the point ξ to the value s, and if $\alpha_n=O(1/n)$, then*

(1) $$\tfrac{1}{2}[s_n(\xi+\alpha_n)+s_n(\xi-\alpha_n)]-(s_n(\xi)-s)\cos n\alpha_n\to s.$$

---

[1] Fejér [3], Rogosinski [2].
[2] Du Bois-Reymond [2], Fejér [3].
[3] Rogosinski [3], [4].
[4] not necessarily a Fourier series.

Abel's transformation shows that

$$(2)\quad \tfrac{1}{2}[s_n(\xi+\alpha_n)+s_n(\xi-\alpha_n)] = \sum_{k=0}^{n-1} s_k(\xi)\,\Delta\cos k\alpha_n + s_n\cos n\alpha_n.$$

Here we have a linear transformation of $\{s_n(\xi)\}$, and the reader will verify that Toeplitz's conditions (§ 3.1) are satisfied. In particular, the condition (iii) of Toeplitz follows from the inequality $|\Delta\cos k\alpha_n| \leqslant \alpha_n = O(1/n)$.

This completes the proof of (i). Making Abel's transformation once more, we obtain, for the left-hand side of (2), the expression

$$(3)\quad \sum_{k=0}^{n-2}(k+1)\,\sigma_k\Delta^2\cos k\alpha_n + \sigma_{n-1}\,n\Delta\cos(n-1)\,\alpha_n + s_n\cos n\alpha_n,$$

where $\sigma_k=\sigma_k(\xi)$ are the first arithmetic means of the series considered. This expression without its last term is a linear transformation of $\{\sigma_n\}$. Toeplitz's conditions (i) and (iii) are again satisfied. Supposing, in particular, that $s_0=s_1=s_2=\ldots=1$, we find that the sum of the coefficients of σ_k in (3) is equal to $(1-\cos n\alpha_n)$. It follows that the expression (3) deprived of its last term and divided by $1-\cos n\alpha_n$ tends to s, and this is just (1). As a corollary we obtain

If 8.11(3a) *is a* $\mathfrak{S}[f]$, ξ *a point of continuity of* f, *and* p *any fixed odd number, then* $\tfrac{1}{2}[s_n(\xi+p\pi/2n)+s_n(\xi-p\pi/2n)]\to f(\xi)$. *This relation holds uniformly in any interval of continuity of* f.

8.62. We know that, if ξ is a point of continuity of f, then $|\sigma_k(\xi+h)-f(\xi)|<\varepsilon$ for $k>\nu$, $|h|<\delta$ [1]). Hence, for any sequence $\{h_n\}\to 0$, we have $|\sigma_k(\xi+h_n)-f(\xi)|\leqslant\varepsilon_k$, $k\leqslant n<\infty$, where $\varepsilon_k\to 0$. It follows that, if $\sigma_k=\sigma_k(\xi+h_n)$, $1\leqslant k<n$, $\alpha_n=\pi/2n$, the expression 8.61(3) is $f(\xi)+o(1)$, and so

If $s_n(x)=s_n(x;f)$ *and* ξ *is a point of continuity of* f, *we have*

$$\tfrac{1}{2}[s_n(\xi+h_n+\pi/2n)+s_n(\xi+h_n-\pi/2n)]\to f(\xi)$$

for every $\{h_n\}\to 0$.

In § 8.11 we learnt that $s_n(x;f)$ may be unbounded in the neighbourhood of a point of continuity of f. The last theorem detects a certain regularity in the behaviour of the curves $y=s_n(x)$: for $|x-\xi|<\varepsilon$, the arithmetic mean of the values of $s_n(x)$ at the

---

[1]) See footnote [1]) on p. 52.

ends of intervals of length π/n differs very little from $f(\xi)$, and the less the smaller ε and $1/n$ are.

8.7. Cramér's theorem. We shall now study Gibbs's phenomenon for the method (C, r). From the inequality 3.22(1) we deduce that Fejér's sums cannot present Gibbs's phenomenon. Moreover it is easy to see that, if this phenomenon does not exist for a value r_1 of r, it cannot exist for any larger value of r. For, if $\sigma_n^r(x)$ denote the Cesàro means for $\mathfrak{S}[f]$ and if we have $m-\varepsilon \leqslant \sigma_n^{r_1}(x) \leqslant M+\varepsilon$ for $|x-\xi| \leqslant \eta$, $n > n_0$, and if $r > r_1$, then $m - 2\varepsilon \leqslant \sigma_n^r(x) \leqslant M + 2\varepsilon$ for $|x-\xi| \leqslant \eta$, $n \geqslant n_1$ (§ 3.13). It is therefore sufficient to consider the case $0 < r < 1$.

There exists a number $0 < r_0 < 1$ with the following property: If f is simply discontinuous at a point ξ, the (C, r) means $\sigma_n^r(x)$ of $\mathfrak{S}[f]$ present Gibbs's phenomenon at ξ for $r < r_0$, but not for $r \geqslant r_0$ [1]*.*

8.701. It is sufficient to prove the theorem for the series 8.5(1), for which we have the formulae

$$(1)\quad \sigma_n^r(x) = -\tfrac{1}{2}x + \int_0^x K_n^r(t)\,dt, \qquad \sigma_n^r(x) = \tfrac{1}{2}(\pi - x) - \int_x^\pi K_n^r(t)\,dt,$$

where K_n^r denotes the (C, r) kernel. Let us consider first the case $r = 1$. Replacing the denominator $4\sin^2 \frac{1}{2}t$ by t^2, we find, as in § 8.5, that

$$(2)\qquad \sigma_n(x) = -\tfrac{1}{2}x + \int_0^{(n+1)\frac{x}{2}} \frac{\sin^2 t}{t^2}\,dt + R_n(x),$$

where $\sigma_n(x) = \sigma_n^1(x)$, $R_n(x) = O(n^{-1}) = o(1)$ uniformly in x. Since $\sigma_n(x) \to (\pi - x)/2$ for $0 < x < 2\pi$, we obtain from (2) that

$$(3)\qquad \int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \tfrac{1}{2}\pi.$$

From (2) and (3) we deduce the following proposition which will be used presently. *Given any number $l > 0$, there exists an $\varepsilon = \varepsilon(l) > 0$ and an integer $n_0 = n_0(l)$, such that $\sigma_n(x) < \pi/2 - \varepsilon$ for $0 \leqslant x \leqslant l/n$, $n > n_0$.*

---

[1]) Cramér [1]. Gronwall [2] showed that $r_0 = 0.4395516\ldots$

8.702. Next we require a formula for $K_n^r(t)$. Such a formula was found in § 3.3(3). Applying Abel's transformation to the last term of it, we find that $K_n^r(t)$ is equal to

$$\Im\left\{\frac{e^{i(n+1/2)t}}{2A_n^r \sin\frac{1}{2} t}\left[\frac{1}{(1-e^{-it})^r} - A_{n+1}^{r-1}\frac{e^{-i(n+1)t}}{1-e^{-it}} - \sum_{\nu=n+1}^{\infty} A_{\nu+1}^{r-2}\frac{e^{-i(\nu+1)t}}{1-e^{-it}}\right]\right\} =$$

$$= \frac{1}{A_n^r}\cdot\frac{\sin\left[(n+\frac{1}{2}+\frac{1}{2}r)t - \frac{1}{2}\pi r\right]}{(2\sin\frac{1}{2} t)^{r+1}} + \frac{r}{n+1}\cdot\frac{1}{(2\sin\frac{1}{2} t)^2} + \frac{\theta}{n^2}\cdot\frac{8r(1-r)}{(2\sin\frac{1}{2} t)^3}\ [1],$$

where $|\theta| \leqslant 1$ (see § 1.22). Integrating this expression over (x, π) and applying the second mean-value theorem to the first integral, we obtain from the second equation 8.701(1) that $\sigma_n^r(x)$ is equal to

$$(1) \qquad \tfrac{1}{2}(\pi - x) - \frac{r}{n+1}\,\tfrac{1}{2}\operatorname{ctg}\tfrac{1}{2}x + \frac{2\theta_1}{nA_n^r(2\sin\frac{1}{2}x)^{r+1}} + \frac{B}{n^2x^2},$$

where $|\theta_1| \leqslant 1$, and $|B|$ is less than an absolute constant.

It was implicitly proved in § 3.12 that there exists an absolute constant C such that $A_n^r \geqslant Cn^r$ if $n \geqslant 1$, $0 \leqslant r \leqslant 1$. This shows that, if nx is large, of the last three terms in (1) the first is the largest in absolute value. Therefore there exists a number l such that $|\sigma_n^r(x)| \leqslant \pi/2$ for $l/n \leqslant x \leqslant \pi$, $1/2 \leqslant r \leqslant 1$, $n \geqslant n_1$.

Now we will show that, if $1 - r$ is small enough, we have $|\sigma_n^r(x)| \leqslant \pi/2$ for $0 \leqslant x \leqslant l/n$. Taking into account the inequality $A_k^r/A_n^r \geqslant A_k^s/A_n^s$, which is true for $0 \leqslant k \leqslant n$, $-1 < r < s$, we find that $|\sigma_n^r(x) - \sigma_n^s(x)|$ is less than

$$(2) \qquad \sum_{\nu=1}^{n}\left(\frac{A_{n-\nu}^r}{A_n^r} - \frac{A_{n-\nu}^s}{A_n^s}\right)\left|\frac{\sin \nu x}{\nu}\right| \leqslant x\left[\frac{A_n^{r+1}}{A_n^r} - \frac{A_n^{s+1}}{A_n^s}\right] = \frac{nx(s-r)}{(r+1)(s+1)}.$$

If $s = 1$, the last expression is less than $\frac{1}{2}nx(1-r)$, and so it is sufficient to take r such that $\frac{1}{2}(1-r)\,l < \varepsilon(l)$ (§ 8.701).

8.703. We have proved that, if r is sufficiently near to 1, σ_n^r cannot present Gibbs's phenomenon. To show that, if $r > 0$ is small enough, Gibbs's phenomenon does occur, we consider the expression $|\sigma_n^r(x) - s_n(x)|$ which, in view of the inequality 8.702(2), is less than $xnr/(r+1)$. Since $s_n(\pi/n) \to \varphi(\pi) > \pi/2$ (§ 8.5), we conclude that Gibbs's phenomenon certainly occurs if we have $\pi r/(r+1) < \varphi(\pi) - \pi/2$.

---

[1] For a different proof, based on complex integration, of this formula, see Kogbetliantz [1].

8.704. In the previous sections we established the existence of a number r_0, $0 < r_0 < 1$, such that for any $r > r_0$ we have Gibbs's phenomenon, whereas for $r < r_0$ we have not. It remains only to show that for $r = r_0$ the phenomenon does not occur.

Let r_1 be any positive number less than r_0. From the formula 8.702(1) for σ_n^r we see that there is a number l_1 such that $|\sigma_n^r(x)| \leqslant \frac{1}{2}\pi$ for $r_1 \leqslant r \leqslant 1$, $l_1/n \leqslant x \leqslant \pi$. From the inequality 8.702(2) for $|\sigma_n^r - \sigma_n^s|$ we see that $\sigma_n^r(x)$ is a uniformly continuous function of r in the range $r \geqslant 0$, $0 \leqslant x \leqslant l_1/n$, $n = 1, 2, \dots$ If the Gibbs phenomenon occurs for a value $r > r_1$, that is if there is a sequence $\{x_n\} \to +0$ such that $|\sigma_n^r(x_n)| > \frac{1}{2}\pi + \varepsilon$, then $0 \leqslant x_n \leqslant l_1/n$ and so, if $|s - r|$ is small enough, $|\sigma_n^s(x)| > \frac{1}{2}\pi + \frac{1}{2}\varepsilon$. This shows that the set of r for which the Gibbs phenomenon occurs is an open set, and the theorem is established.

8.8. Miscellaneous theorems and examples.

1. The Lebesgue constant L_n is equal to

$$\frac{16}{\pi^2}\sum_{\nu=1}^{\infty}\{1 + 1/3 + 1/5 + \dots + 1/[2\nu(2n+1) - 1]\}/(4\nu^2 - 1).$$

From this formula we see that $\{L_n\}$ is an increasing sequence. Szegö [2].

[Consider $\mathfrak{S}[|\sin x|]$ (§ 1.8.2) and the formula

$$(\sin kx)^2/\sin x = \sin x + \sin 3x + \dots + \sin(2k-1)x].$$

2. Theorems 3.5(i) and 3.5(ii) are false for $r = 1$.

[To prove the first part of this assertion, show that $\int_0^\pi \sin t\,|K_n'(t)|\,dt \neq O(1)$, where K_n denotes Fejér's kernel, and apply an argument similar to that of § 8.31. For the second part we refer the reader to Hahn [2]].

3. A series $u_0 + u_1 + \dots$ is said to be summable by Borel's method, or summable B, to sum s, if $e^{-x}\sum_{n=0}^{\infty} s_n x^n/n! \to s$ as $x \to \infty$, where $s_n = u_0 + \dots + u_n$. Show that

(i) If a series is convergent, it is summable B to the same sum.

(ii) A power series may be summable B outside its circle of convergence, so that the method B is rather strong. Nevertheless,

(iii) There exist continuous functions with Fourier series non-summable B at some points. Moore [1].

(iv) If $[f(x_0 + h) - f(x_0)]\log 1/|h| \to 0$ with h, $\mathfrak{S}[f]$ is summable B at the point x_0, to the value $f(x_0)$. Hardy and Littlewood [2].

[Ad (i): apply Toeplitz's theorem (§ 3.1). Ad (ii): the series $1+z+z^2+\dots$ is summable B for $\Re z < 1$. To prove (iii) it is sufficient to observe that the Lebesgue constants corresponding to the method B form an unbounded *function*. These constants, which are equal to

$$\frac{2}{\pi}\int_0^\pi e^{-x(1-\cos t)} \frac{|\sin(x\sin t+\frac{1}{2}t)|}{2\sin\frac{1}{2}t}\, dt,$$

are of order $\log x$. Propositions (ii) and (iii) show that the methods B and (C,k), $k>0$, are not comparable]

4. Consider a sequence $p_0, p_1, \dots$ of positive numbers, with the properties that $P_n = p_0+p_1+\dots+p_n \to \infty$, $p_n/P_n \to 0$. A series $u_0+u_1+\dots$ is said to be summable by Nörlund's method corresponding to $\{p_\nu\}$, or summable $N\{p_\nu\}$, to sum s, if

$$\sigma_n = (s_0 p_n + s_1 p_{n-1} + \dots + s_n p_0)/P_n = (u_0 P_n + \dots + u_n P_0)/P_n \to s$$

as $n\to\infty$. If $P_n = A_n^\alpha$, $\alpha>0$, we obtain, as a special case, Cesàro's method of summation (§ 3.11). Show that

(i) If Σu_n converges, it is summable $N\{p_\nu\}$ to the same sum.

(ii) If $0 < p_0 \leqslant p_1 \leqslant \dots$ and if Σu_n is summable $(C,1)$, it is also summable $N\{p_\nu\}$ to the same sum. Tamarkin, *Fourier series*, p. 156.

5. Let $p_\nu > p_{\nu+1} \to 0$, $P_\nu \to \infty$. A necessary and sufficient condition that the method $N\{p_\nu\}$ should sum $\mathfrak{S}[f]$, to the value $f(x)$, at every point of continuity of f, is that the sequence

$$\lambda_n = P_n^{-1}\sum_{\nu=1}^{n}\frac{P_\nu}{\nu}$$

should be bounded Hille and Tamarkin [1$_2$], Tamarkin, *Fourier Series*, 190.

[In the first place we show that, if $\lambda_n = O(1)$, then the $N\{p_\nu\}$ kernel is quasi-positive (§ 3.201). Conditions (ii) and (iii) follow immediately. To prove (i') we argue as in § 3.3 and obtain, for the kernel, the expression

$$\frac{\sin(n+1)t}{2\sin\frac{1}{2}t}P_n^{-1}\sum_{\nu=0}^{n}p_\nu\cos(\nu+\tfrac{1}{2})t - \frac{\cos(n+1)t}{2\sin\frac{1}{2}t}P_n^{-1}\sum_{\nu=0}^{n}p_\nu\sin(\nu+\tfrac{1}{2})t = U_n - V_n.$$

Applying Abel's transformation to V_n, and denoting by K_n Fejér's kernel, we find that $\mathfrak{M}[V_n;\,0,\pi]$ does not exceed

$$P_n^{-1}\int_0^\pi\Big\{(n+1)p_nK_n+\sum_{\nu=0}^{n-1}(\nu+1)\Delta p_\nu K_\nu\Big\}dt = \tfrac{1}{2}\pi P_n^{-1}\Big\{(n+1)p_n+\sum_{\nu=0}^{n-1}(\nu+1)\Delta p_\nu\Big\}.$$

The expression in curly brackets is equal to P_n, so that $\mathfrak{M}[V_n; 0, \pi] \leqslant \tfrac{1}{2}\pi$, and everything depends on the behaviour of $\mathfrak{M}[U_n; 0, \pi] = \mathfrak{M}[U_n; 0, 1/n] + \mathfrak{M}[U_n; 1/n, \pi]$. It is easy to see that $U_n = O(n)$ in the interval $0 \leqslant t \leqslant 1/n$, so that $\mathfrak{M}[U_n; 0, 1/n] = O(1)$. Now Abel's transformation gives, for U_n, the value

$$(1) \qquad \frac{\sin(n+1)t}{P_n}\left\{p_n \frac{\sin(n+1)t}{4\sin^2\tfrac{1}{2}t} + \sum_{\nu=0}^{n-1} \Delta p_\nu \frac{\sin(\nu+1)t}{4\sin^2\tfrac{1}{2}t}\right\}.$$

Observing that $\int_{1/n}^{\pi} \frac{|\sin(\nu+1)t|}{4\sin^2\frac{1}{2}t}dt = \int_{1/n}^{1/\nu} + \int_{1/\nu}^{\pi} \leqslant (\nu+1)\int_{1/n}^{1/\nu} \frac{t\,dt}{4\sin^2\frac{1}{2}t} + \int_{1/\nu}^{\pi} \frac{dt}{4\sin^2\frac{1}{2}t} < A(\nu+1)\log(n/\nu) + B(\nu+1)$, $\nu \geqslant 1$, where A and B are constants, we see that the absolute value of (1) integrated over $(1/n, \pi)$ gives less than

$$\frac{B}{P_n}\{(n+1)p_n + \sum_{\nu=0}^{n-1}(\nu+1)\Delta p_\nu\} + \frac{A}{P_n}\left\{\sum_{\nu=1}^{n-1}\Delta p_\nu \cdot (\nu+1)\log(n/\nu)\right\} + O(P_n^{-1}\log n).$$

Here the first term is equal to B. Making Abel's transformation, we see that the second term is equal to

$$\frac{2Ap_1}{P_n}\log n + \frac{A}{P_n}\left\{\sum_{\nu=2}^{n} p_\nu \log(n/\nu)\right\} + \frac{A}{P_n}\left\{\sum_{\nu=2}^{n} \nu p_\nu \log\left(1 - \frac{1}{\nu}\right)\right\} = A_n + B_n + C_n.$$

It is not difficult to verify that the condition $\lambda_n = O(1)$ implies $\log n = O(P_n)$, i. e. $A_n = O(1)$. Since $\log(1 - 1/\nu) \simeq -1/\nu$, we obtain $C_n = O(1)$. Applying Abel's transformation, we see that $B_n = O(\lambda_n) + O(P_n^{-1}\log n) = O(1)$. Hence $\mathfrak{M}[U_n; 1/n, \pi] = O(1)$, $\mathfrak{M}[U_n; 0, \pi] = O(1)$, and the first half of the theorem is established.

To prove the second half, it is sufficient to show that, if $\mathfrak{M}[U_n] = O(1)$, then $\lambda_n = O(1)$. Applying Abel's transformation to U_n and observing that $|\sin(n+1)t| \geqslant \sin^2(n+1)t$, we see that the relation $\mathfrak{M}[U_n; 0, \pi] = O(1)$ implies

$$(2) \qquad P_n^{-1}\int_0^{\pi} \sin^2(n+1)t\left\{P_n \frac{\cos(n+\frac{1}{2})t}{2\sin\frac{1}{2}t} + \sum_{\nu=0}^{n-1} P_\nu \sin(\nu+1)t\right\}dt = O(1).$$

It is not difficult to see that the integral, extended over $(0, \pi)$, of the function $\sin^2(n+1)t \cdot \cos(n+\tfrac{1}{2})t/2\sin\tfrac{1}{2}t$ is bounded. Hence, using the equation $2\sin^2(n+1)t = 1 - \cos 2(n+1)t$, and the fact that the integral over $(0, \pi)$ of $\sin(\nu+1)t\cos 2(n+1)t$ is $O(1/n)$ for $0 \leqslant \nu < n$, we see that (2) may be written

$$\tfrac{1}{2}P_n^{-1}\int_0^{\pi}\left\{\sum_{\nu=0}^{n-1} P_\nu \sin(\nu+1)t\right\}dt + R = O(1),$$

where $R = O(P_0 + P_1 + \dots + P_{n-1})/n P_n = O(1)$ From this we obtain $\lambda_n = O(1)$].

6. The partial sums $d_n(x)$ of the series $\sin x + \frac{1}{2}\sin 2x + \dots$ are positive for $0 < x < \pi$. Jackson [1]; Landau [1].

[Suppose that the theorem has been established for $n-1$ and that $d_n(x)$, $0 \leqslant x \leqslant \pi$, attains its minimum at a point x_0, $0 < x_0 < \pi$. Since

$$d_n'(x_0) = [\sin(n+\tfrac{1}{2})\,x_0 - \sin\tfrac{1}{2}x_0]/2\sin\tfrac{1}{2}x_0 = 0,$$

we obtain that $\sin(n+\frac{1}{2})\,x_0 = \sin\frac{1}{2}x_0$ and so also $|\cos(n+\frac{1}{2})\,x_0| = \cos\frac{1}{2}x_0$. This shows that $\sin nx_0 = \sin(n+\frac{1}{2})\,x_0\cos\frac{1}{2}x_0 - \cos(n+\frac{1}{2})\,x_0\sin\frac{1}{2}x_0 \geqslant 0$, $d_n(x_0) \geqslant d_{n-1}(x_0)$, which is impossible since the theorem is true for $n-1$].

CHAPTER IX.

Further theorems on Fourier coefficients. Integration of fractional order.

9.1. Remarks on the theorems of Hausdorff-Young and F. Riesz. It has been proved in Chapter IV that, for any complex function $f(t)$ with Fourier coefficients c_n, we have

$$(1) \qquad \frac{1}{2\pi}\int_0^{2\pi} |f(t)|^2\, dt = \sum_{n=-\infty}^{+\infty} |c_n|^2.$$

This formula contains two propositions: (i) If $f \in L^2$, the series on the right converges to the sum equal to the integral on the left (*Parseval*), (ii) If c_n is an arbitrary sequence such that $\Sigma |c_n|^2$ converges, there is an $f \in L^2$ with complex Fourier coefficients c_n satisfying (1) *(Riesz-Fischer)*. It is natural to inquire how far these results can be extended to exponents other than 2. It appears that such extensions are possible, but only partly. Here we shall only state the results and make a few remarks about them. Complete proofs will be given in § 9.3.

Given any function $f(t)$, $0 \leqslant t \leqslant 2\pi$, and any sequence $\{c_n\}$, $-\infty < n < +\infty$, we write

$$\mathfrak{A}_r[f] = \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(t)|^r\, dt\right\}^{1/r}, \qquad \mathfrak{N}_r[c] = \left\{\sum_{n=-\infty}^{+\infty} |c_n|^r\right\}^{1/r}.$$

We assume that f and c_n may take complex values. Using the letters p and q, we shall suppose, unless a statement to the contrary is made, that $1 < p \leqslant 2 \leqslant q < \infty$. For any $r \geqslant 1$ we define r' by the condition $1/r + 1/r' = 1$, so that p' is a q, q' is a p.

The following theorem is due to Hausdorff and Young[1]).

(a) *If* $f \in L^p$ *and* c_n *are the Fourier coefficients of* f, *then* $\mathfrak{N}_{p'}[c]$ *is finite and* $\mathfrak{N}_{p'}[c] \leqslant \mathfrak{A}_p[f]$.

(b) *Given any sequence of numbers* c_n, $-\infty < n < +\infty$, *such that* $\mathfrak{N}_p[c] < \infty$, *there is a function* $f \in L^{p'}$ *with Fourier coefficients* c_n *and such that* $\mathfrak{A}_{p'}[f] \leqslant \mathfrak{N}_p[c]$.

Theorem (a) is an extension of Parseval's theorem, the sign = being replaced by $\leqslant$; Theorem (b) is an extension of the Riesz-Fischer theorem. In both (a) and (b) the argument goes from p to p', i. e. from the smaller number to the larger. The theorem would be false if we replaced p by q. For (i) there is a continuous function f such that $\mathfrak{N}_p[c] = \infty$ for every $p < 2$ (§§ 5.33, 5.61), (ii) there exist trigonometrical series which are not Fourier series and have coefficients c_n such that $\mathfrak{N}_q[c] < \infty$ for every $q > 2$; the series $\Sigma\, n^{-1/2} \cos 2^n x$ is an instance in point (§ 5.4). Roughly speaking, the theorems of Parseval and of Riesz-Fischer are the best: we can neither strengthen the thesis of the former, nor weaken the hypothesis of the latter.

The reader will observe that between the two parts of the Hausdorff-Young theorem there is a certain analogy. The second part may be obtained from the first if the function f, depending on the variable t, is replaced by the function c depending on the variable n, integration is replaced by summation and vice versa. This fact is explained by the theory of Fourier integrals, where both parts of the theorem corresponding to that of Hausdorff-Young coincide. The analogy just stated can be detected in various theorems of the theory of Fourier series and is an important guide in the search of new results. We shall not investigate this subject systematically.

9.11. The Hausdorff-Young theorem can be extended to general systems of complex functions $\varphi_1, \varphi_2, \ldots$ which are orthogonal, normal, and uniformly bounded ($|\varphi_n| \leqslant M$, $n = 1, 2, \ldots$) in an interval (a, b). Let us consider an arbitrary function $f(t)$, $a \leqslant t \leqslant b$, and an arbitrary sequence of numbers $c_1, c_2, \ldots$, and put

$$\mathfrak{M}_r[f] = \mathfrak{M}_r[f;\, a, b] = \left(\int_a^b |f|^r\, dt\right)^{1/r}, \quad \mathfrak{N}_r[c] = \left(\sum_{n=1}^{\infty} |c_n|^r\right)^{1/r}.$$

1) Young [12], [13] proved the theorem in the case $p' = 2k$, $k = 1, 2, \ldots$ The general result is due to Hausdorff [2].

F. Riesz's extension of the Hausdorff-Young theorem may be stated as follows [1]).

(a) *If* $f \in L^p(a, b)$ *and if* c_n *are the Fourier coefficients of* f *with respect to* $\{\varphi_n\}$, *then* $\mathfrak{N}_{p'}[c]$ *is finite and* $\mathfrak{N}_{p'}[c] \leqslant M^{(2-p)/p} \mathfrak{M}_p[f]$.

(b) *If, for a given* $\{c_n\}$, *we have* $\mathfrak{N}_p[c] < \infty$, *there is an* $f \in L^{p'}(a, b)$ *whose Fourier coefficient with respect to* φ_n *is* c_n, $n = 1, 2, \ldots$, *and such that* $\mathfrak{M}_{p'}[f] \leqslant M^{(2-p)/p} \mathfrak{N}_p[c]$.

Applying this theorem to the system of functions e^{ikx}, $k = 0, \pm 1, \ldots$, $0 \leqslant x \leqslant 2\pi$, we obtain the Hausdorff-Young theorem.

9.12. The Hausdorff-Young theorem will be established, as a corollary of F. Riesz's theorem, in § 9.3. Here we give an independent proof of the former theorem in the case $p' = 2k$, i. e. $p = 2k/(2k-1)$, $k = 1, 2, \ldots$ This case is fairly easy and, what is more important, in certain interesting applications of the Hausdorff-Young theorem it suffices entirely.

Given an $f \in L$, we put $f(x) = f_1(x)$ and

$$f_i(x) = \frac{1}{2\pi} \int_0^{2\pi} f_{i-1}(x+t) f(-t)\, dt, \qquad i = 2, 3, \ldots$$

From Theorem 2.11 we see that, if c_n are the Fourier coefficients of f, those of f_i are c_n^i. From § 4.16(ii) we obtain, by induction, that, if $\alpha_i > 0$, $i = 1, 2, \ldots, j$, $\alpha_1 + \alpha_2 + \ldots + \alpha_j < 1$, then

$$\mathfrak{A}_{1/(1-\alpha_1-\ldots-\alpha_j)}[f_j] \leqslant \prod_{i=1}^{j} \mathfrak{A}_{1/(1-\alpha_i)}[f].$$

Putting $j = k$, $\alpha_1 = \alpha_2 = \ldots = 1/2k$, and supposing that $f \in L^{2k/(2k-1)}$, we obtain $\mathfrak{A}_2[f_k] \leqslant \mathfrak{A}^k_{2k/(2k-1)}[f]$. Hence, observing that the Fourier coefficients of f_k are c_n^k and applying Parseval's theorem, we have

$$\mathfrak{A}_{2k/(2k-1)}[f] \geqslant \mathfrak{A}_2^{1/k}[f_k] = \mathfrak{N}_2^{1/k}[c^k] = \mathfrak{N}_{2k}[c],$$

i. e. $\mathfrak{A}_{2k/(2k-1)}[f] \geqslant \mathfrak{N}_{2k}[c]$; this is just Theorem 9.1(a) for $p' = 2k$.

9.121. Theorem 9.1(b) may be obtained by a similar argument, using, instead of the results of § 4.16, analogous results for sequences. We prefer to follow a different way and to deduce Theorem 9.1(b) from Theorem 9.1(a), or, more generally, Theorem 9.11(b) from Theorem 9.11(a).

Suppose that $\mathfrak{N}_p[c] < \infty$, and let $f_n = c_1\varphi_1 + \ldots + c_n\varphi_n$, $n = 1, 2, \ldots$ For every function g with Fourier coefficients $d_1, d_2, \ldots$ we have

---

[1]) F. Riesz [6].

$$\left|\int_a^b \overline{f}_n g\,dx\right| = \left|\sum_1^n \overline{c}_\nu d_\nu\right| \leqslant \left(\sum_1^n |c_\nu|^p\right)^{1/p} \left(\sum_1^n |d_\nu|^{p'}\right)^{1/p'} \leqslant \mathfrak{N}_p[c]\, M^{(2-p)/p}\, \mathfrak{M}_p[g],$$

the last inequality being an application of Theorem 9.11(a). The upper bound of the left-hand side, for all g with $\mathfrak{M}_p[g] \leqslant 1$, is equal to $\mathfrak{M}_{p'}[\overline{f}_n] = \mathfrak{M}_{p'}[f_n]$ (§ 4.7.2), so that

(1) $$\mathfrak{M}_{p'}[f_n] \leqslant M^{(2-p)/p}\, \mathfrak{N}_p[c].$$

Since the inequality $\mathfrak{N}_p[c] < \infty$ implies $\mathfrak{N}_2[c] < \infty$, the series $c_1\varphi_1 + c_2\varphi_2 + \ldots$ is the Fourier series of a function f (§ 4.21). If n tends to ∞ through a particular sequence of values, then $f_n(x) \to f(x)$ almost everywhere (§ 4.2) and, applying Fatou's lemma, we deduce from (1) that $\mathfrak{M}_{p'}[f] \leqslant M^{(2-p)/p}\mathfrak{N}_p[c]$, i. e. Theorem 9.11(b).

In a similar way we could deduce Theorem 9.11(a) from Theorem 9.11(b), so that both theorems are in reality equivalent.

9.2. M. Riesz's convexity theorems[1]. Consider a system of numbers a_{jk}, $1 \leqslant j \leqslant m$, $1 \leqslant k \leqslant n$, and the linear forms $X_j = a_{j1}x_1 + a_{j2}x_2 + \ldots + a_{jn}x_n$, $j = 1, 2, \ldots, m$, of the variables $x_1, x_2, \ldots, x_n$. Let $M_{\alpha\beta}$ denote the upper bound of the expression $(\sigma_1 |X_1|^{1/\beta} + \ldots + \sigma_m |X_m|^{1/\beta})^\beta$ for the values of $x_1, x_2, \ldots x_n$, satisfying the inequality $(\rho_1 |x_1|^{1/\alpha} + \ldots + \rho_n |x_n|^{1/\alpha})^\alpha \leqslant 1$, that is

(1) $$M_{\alpha\beta} = \operatorname*{Max}_{x_1, \ldots, x_n} \left(\sum_{j=1}^m \sigma_j |X_j|^{1/\beta}\right)^\beta \bigg/ \left(\sum_{k=1}^n \rho_k |x_k|^{1/\alpha}\right)^\alpha, \qquad (\alpha, \beta > 0),$$

where σ_j and ρ_k are arbitrary but fixed positive numbers. It is easy to see that the maximum is attained for every $\alpha, \beta > 0$.

$M_{\alpha\beta}$ is a multiplicatively convex function of the variables α, β in the triangle (Δ) $0 \leqslant \alpha \leqslant 1$, $0 \leqslant \beta \leqslant \alpha$.

We mean by this that on an arbitrary segment l which lies entirely in Δ, $M_{\alpha\beta}$, considered as a function of a point, is multiplicatively convex (§ 4.14). To show this it is sufficient to prove that, for every point $P(\alpha, \beta)$ lying inside l, there exist on l, arbitrarily near P, two points $P_1(\alpha_1, \beta_1)$ and $P_2(\alpha_2, \beta_2)$, such that $P = t_1 P_1 + t_2 P_2$, $t_1 > 0$, $t_2 > 0$, $t_1 + t_2 = 1$, and that $M_{\alpha\beta} \leqslant M_{\alpha_1\beta_1}^{t_1} M_{\alpha_2\beta_2}^{t_2}$ [2].

---

[1] M. Riesz [3]; Paley [2].

[2] If a function $y = \varphi(x)$ is not convex, there is an arc $y = \varphi(x)$, $x_1 < x < x_2$, lying totally above its chord $y = l(x)$, $x_1 < x < x_2$. Let x_0 be the largest value of the argument x, $x_1 < x < x_2$, for which $\varphi(x) - l(x)$ attains its maximum. Then, for any numbers x_1' and x_2' such that $x_1 < x_1' < x_0 < x_2' < x_2$, the point $(x_0, \varphi(x_0))$ lies above the chord joining $(x_1', \varphi(x_1'))$ and $(x_2', \varphi(x_2'))$.

Since $M_{\alpha\beta}$ is a continuous function of α, β [1]), we may restrict ourselves to the case of l lying entirely inside Δ. We may also suppose that l is not parallel to the β-axis.

Let us fix α, β, and put $a = 1/\alpha$, $b = 1/\beta$. Let $x_1, x_2, \dots, x_n$ be a system of values for which the maximum in (1) is attained; $y_1, y_2, \dots, y_n$ denotes a system of numbers which will be defined presently, and $Y_1, Y_2, \dots, Y_m$ are the corresponding values of the linear forms. The expression

$$(2) \qquad (\Sigma\, \sigma_j\, |X_j + \varepsilon\, Y_j|^b)^\beta / (\Sigma\, \rho_k\, |x_k + \varepsilon\, y_k|^a)^\alpha,$$

considered as a function of ε, attains its maximum for $\varepsilon = 0$. Let $x = x' + ix''$, $y = y' + iy''$. It is easy to see that, if $a > 1$, the expression $|x + \varepsilon y|^a = [(x' + \varepsilon y')^2 + (x'' + \varepsilon y'')^2]^{a/2}$ is a differentiable function of ε, and its derivative at the point $\varepsilon = 0$ is $\Re\, a|x|^{a-1}\overline{(\operatorname{sign} x)}y$. Hence the ratio (2) is also differentiable and, equating its derivative at the point 0 to 0, we obtain the formula

$$(3) \quad \Sigma\sigma_j|X_j|^b/\Sigma\rho_k|x_k|^a = \Re\,\Sigma\sigma_j|X_j|^{b-1}\overline{(\operatorname{sign} X_j)}Y_j / \Re\,\Sigma\rho_k|x_k|^{a-1}\overline{(\operatorname{sign} x_k)}y_k.$$

Let us put $y_k = |x_k|^\lambda \operatorname{sign} x_k$; thence $|x_k| = |y_k|^{1/\lambda}$ and the denominator on the right may be written in the form of a product $(\Sigma\, \rho_k\, |x_k|^{a-1+\lambda})^{\theta_1} (\Sigma\, \rho_k\, |y_k|^{(a-1+\lambda)/\lambda})^{\theta_2}$, where the numbers $\lambda, \theta_1 > 0$, $\theta_2 > 0$, $\theta_1 + \theta_2 = 1$, will be fixed presently. Let us represent the coefficient of σ_j on the right in (3) in the form $|X_j|^{b-a} \cdot |X_j|^{a-1} \cdot \overline{(\operatorname{sign} X_j)}\, Y_j$. Applying Hölder's inequality with exponents k, k_1, k_2, where $1/k + 1/k_1 + 1/k_2 = 1$, $k = b/(b-a)$, we obtain from (3)

$$\frac{\Sigma\, \sigma_j\, |X_j|^b}{\Sigma\, \rho_k\, |x_k|^a} \leqslant \frac{(\Sigma\, \sigma_j\, |X_j|^b)^{(b-a)/b}\, (\Sigma\, \sigma_j\, |X_j|^{(a-1)k_1})^{1/k_1}\, (\Sigma\, \sigma_j\, |Y_j|^{k_2})^{1/k_2}}{(\Sigma\, \rho_k\, |x_k|^{1/\alpha_1})^{\theta_1}\, (\Sigma\, \rho_k\, |y_k|^{1/\alpha_2})^{\theta_2}}.$$

Here $\alpha_1 = 1/(a-1+\lambda)$, $\alpha_2 = \lambda/(a-1+\lambda)$, whence $(a-1)\alpha_1 + \alpha_2 = 1$, that is $(1-\alpha)\,\alpha_1 + \alpha\alpha_2 = \alpha$. Let us put

---

[1]) Considering separately the cases $\alpha > 0$ and $\alpha = 0$, we prove that the denominator in (1) is a continuous function of α, $x_1, \dots, x_n$ in the range $\alpha \geqslant 0$, $x_1, \dots x_n$ arbitrary. Hence, denoting the ratio in (1) by $f(\alpha, \beta, x_1, \dots, x_n)$, we see that f is continuous in the range $\alpha \geqslant 0$, $\beta \geqslant 0$, $|x_1|^2 + \dots + |x_n|^2 \neq 0$. Since we may plainly define $M_{\alpha\beta}$ as the maximum of f on the 'sphere' $(S)\; |x_1|^2 + \dots + |x_n|^2 = 1$, and since f is uniformly continuous on S, $M_{\alpha\beta}$ is a continuous function of α, β. It must be remembered that, if $\alpha = 0$, the denominator of (1) is equal to $\operatorname{Max}(|x_1|, |x_2|, \dots |x_n|)$ (§ 4.12).

$$\theta_1 = \alpha_1 (a-1), \quad \theta_2 = \alpha_2, \quad (a-1) k_1 = 1/\beta_1, \quad k_2 = 1/\beta_2,$$

so that $\theta_1+\theta_2=1$. The relation $1/k_1+1/k_2=a/b$ gives $(a-1)\beta_1+\beta_2=a/b$, that is $(1-\alpha)\beta_1+\alpha\beta_2=\beta$. From the last inequality we obtain easily

$$(4) \qquad M_{\alpha\beta} \leqslant M_{\alpha_1\beta_1}^{1-\alpha} M_{\alpha_2\beta_2}^{\alpha}.$$

The formulae $(1-\alpha)\alpha_1+\alpha\alpha_2=\alpha$, $(1-\alpha)\beta_1+\alpha\beta_2=\beta$ show that (α,β) lies on the segment l' joining (α_1,β_1) and (α_2,β_2). If for λ we take a value sufficiently near to 1, it follows from the definitions that α_1 and α_2 will be near α. When k_2 runs from the smallest possible value, viz. $b/a=\alpha/\beta$ corresponding to $k_1=\infty$, to infinity, then β_2 varies from β/α to 0. Since $\beta/\alpha>\beta$ and since α_2 is as near α as we please, we find, taking for k_2 a suitable value, that the point (α_2,β_2) lies on l. Then the directions of l and l' coincide, and the formula (4) shows $M_{\alpha\beta}$ to be a multiplicatively convex function on l. This proves the theorem.

9.21. So far, whenever we spoke of the Stieltjes integral $\int_a^b f(x)\,d\varphi(x)$, we understood this integral in the Stieltjes-Riemann sense. Now we shall introduce the Stieltjes-Lebesgue integral, restricting ourselves to the case when $\varphi(x)$ is a non-decreasing function.

Let $y=\varphi(x)$ be a function non-decreasing in an interval $a\leqslant x\leqslant b$, and let $\psi(y)$, $c\leqslant y\leqslant d$, be the inverse function, where $c=\varphi(a)$, $d=\varphi(b)$. If $\varphi(x)$ takes a constant value y_0 for $\alpha<x<\beta$, we assign to $\psi(y_0)$ any value from the interval (α,β). If $\varphi(x_0-0)<\varphi(x_0+0)$, we put $\psi(y)=x_0$ for y belonging to the interval $(\varphi(x_0-0),\varphi(x_0+0))$. Let $f(x)=f(\psi(y))=g(y)$. If $g(y)$ is integrable over (c,d), we say that f is integrable with respect to φ over (a,b) and define the integral by the formula

$$(1) \qquad \int_a^b f(x)\,d\varphi(x) = \int_{\varphi(a)}^{\varphi(b)} g(y)\,dy \text{ [1]}.$$

Since the number of stretches of invariability of $\varphi(x)$ is at most enumerable, the values of $\psi(y)$ corresponding to these stretches have no influence upon the value of the integral.

A set E of points x is said to be of measure 0 with respect to φ, if the variation of φ over E is equal to 0, that is if we can cover E by a finite or enumerable system of intervals (a_i,b_i) such that $\Sigma\{\varphi(b_i)-\varphi(a_i)\}$ is arbitra-

---

[1] For a detailed discussion we refer the reader to Lebesgue's, *Leçons sur l'intégration.*

rily small. This is the same thing as to say that the set E on the x-axis is transformed by the function $y = \varphi(x)$ into a set of ordinary measure 0 on the y-axis[1]). It is plain that, if E is of measure 0 with respect to φ, the left-hand side of (1) is not affected if we change the values of $f(x)$ in E. The function f may even be undefined in E. If $f(x) = f_1(x)$ outside E, we shall not distinguish f from f_1.

A function $\varphi(x)$, $a \leqslant x \leqslant b$, is called a *step*-function if (a, b) can be broken up into a finite number of intervals in the interior of which $\varphi(x)$ is constant. If $x_1, x_2, \ldots, x_k$ are the points of discontinuity of a step-function φ, then $\int_a^b f(x)\,d\varphi(x) = \Sigma\, \sigma_i f(x_i)$, where $\sigma_i = \varphi(x_i + 0) - \varphi(x_i - 0)$. For such functions a set is of measure 0 with respect to φ if it does not contain any of the points x_i. It can be proved that, if $\varphi(x)$ is absolutely continuous and non-decreasing, the left-hand side of (1) is equal to $\int_a^b f(x)\varphi'(x)\,dx$, but we shall not require this result, except in very special cases such as $\varphi(x) = -1/x$.

As regards the applications we have in view, the Stieltjes-Lebesgue integration is not really necessary and we could work with Lebesgue's definition of an integral. The use of the Lebesgue-Stieltjes integral has however certain advantages, the chief of them being that it enables us to treat series ($\varphi(x) =$ a step function) and integrals ($\varphi(x) = x$) in the same way, so that the arguments and results can be stated in a concise form.

We shall denote by $L^{r,\varphi} = L^{r,\varphi}(a, b)$ the class of functions $f(x)$ such that $|f(x)|^r$ is integrable with respect to $\varphi(x)$ over (a, b), and write

$$\mathfrak{M}_r[f] = \mathfrak{M}_{r,\varphi}[f] = \mathfrak{M}_{r,\varphi}[f; a, b] = \left\{\int_a^b |f(x)|^r\, d\varphi(x)\right\}^{1/r}.$$

From (1) and §§ 4.12, 4.13, we deduce the generalized Hölder and Minkowski inequalities

$$\mathfrak{M}[fg] \leqslant \mathfrak{M}_r[f]\,\mathfrak{M}_{r'}[f], \qquad \mathfrak{M}_r[f_1 + f_2] \leqslant \mathfrak{M}_r[f_1] + \mathfrak{M}_r[f_2], \qquad r \geqslant 1,$$

where $\mathfrak{M}_r = \mathfrak{M}_{r,\varphi}$. If f is a step-function, then $\mathfrak{M}_{\infty,\varphi}[f]$ is equal to the upper bound of $|f|$.

Let S denote the class of step-functions $s(x)$, $a \leqslant x \leqslant b$, which vanish in the intervals where $\varphi(x)$ is unbounded. It is plain that such intervals, if they exist, must be extreme intervals.

(i) *The set S is everywhere dense in every class $L^{r,\varphi}$ $1 \leqslant r < \infty$.*

Suppose first that the intervals (a, b) and $(\varphi(a), \varphi(b))$ are both finite, and let $a = a_0 < a_1 < a_2 < \ldots < a_n = b$ be a subdivision of the interval (a, b) such that

[1]) We define the image of a point x as the interval $\varphi(x-0) \leqslant y \leqslant \varphi(x+0)$ of the y-axis.

the points $a_1, a_2, \dots, a_{n-1}$ are points of continuity of φ. Let $l_i = \varphi(a_i) - \varphi(a_{i-1})$; we define a step-function $s(x)$ by the following conditions: if $l_i \neq 0$, we put

(2) $$s(x) = \frac{1}{l_i} \int_{a_{i-1}}^{a_i} f(x)\, d\varphi(x), \quad a_{i-1} < x < a_i, \quad i = 1, 2, \dots, n;$$

if $l_i = 0$, we put $s(x) = 0$ for $a_{i-1} < x < a_i$; in any case $s(b) = s(a_{n-1})$. Applying Hölder's inequality, we obtain that $\mathfrak{M}_{r,\varphi}[s; a_{i-1}, a_i] \leqslant \mathfrak{M}_{r,\varphi}[f; a_{i-1}, a_i]$, and so

(3) $$\mathfrak{M}_{r,\varphi}[s; a, b] \leqslant \mathfrak{M}_{r,\varphi}[f; a, b],$$

an inequality which will be used in a moment.

Now let us consider a sequence of subdivisions of the interval (a, b) such that $\operatorname{Max}(a_i - a_{i-1})$ tends to 0, and the sequence $s_1, s_2, \dots$ of the corresponding functions s. If x_0 is a point of discontinuity of φ, then $s_m(x_0) \to f(x_0)$, i. e. $g_m(y) \to g(y)$ for $\varphi(x_0 - 0) < y < \varphi(x_0 + 0)$, where $g_m(y) = s_m[\psi(y)]$. Let E be the set of the points y which correspond to the intervals of constancy of φ; E is at most enumerable. If y corresponds to a point of continuity of φ and does not belong to E, then $g_m(y) \to g(y)$ provided that $g(y)$ is the derivative, at the point y, of the integral of g. It follows that $g_m(y) \to g(y)$ for almost every y. Hence, if f is bounded,

$$\int_{\varphi(a)}^{\varphi(b)} |g(y) - g_m(y)|^r\, dy \to 0, \quad \text{i. e.} \quad \int_a^b |f(x) - s_m(x)|^r\, d\varphi(x) \to 0.$$

If $f \in L^{r,\varphi}$, we write $f = f' + f''$, where f' is bounded and $\mathfrak{M}_{r,\varphi}[f''] < 1/3\,\varepsilon$. Correspondingly $s_m(x) = s'_m(x) + s''_m(x)$ and

$$\mathfrak{M}_{r,\varphi}[f - s_m] \leqslant \mathfrak{M}_{r,\varphi}[f' - s'_m] + \mathfrak{M}_{r,\varphi}[f''] + \mathfrak{M}_{r,\varphi}[s''_m] < \mathfrak{M}_r[f' - s'_m] + 2/3\,\varepsilon < \varepsilon$$

for m sufficiently large. This shows that $\mathfrak{M}_{r,\varphi}[f - s_m] \to 0$, and (i) is established in the case considered.

To prove (i) in the general case, we again write $f = f' + f''$, where $f' = 0$ outside an interval (a', b') completely interior to (a, b), $f'(x) = f(x)$ in (a', b'), and $\mathfrak{M}_{r,\varphi}[f''] < \frac{1}{2}\varepsilon$. Let $h(x)$ be a step-function vanishing outside (a', b') and such that $\mathfrak{M}_{r,\varphi}[f' - h; a', b'] < \frac{1}{2}\varepsilon$. Then

$$\mathfrak{M}_{r,\varphi}[f - h; a, b] \leqslant \mathfrak{M}_{r,\varphi}[f' - h; a, b] + \mathfrak{M}_{r,\varphi}[f''; a, b] < \varepsilon$$

and this proves the theorem in the general case.

We shall now prove the following result, which will be required in the next section.

(ii) *Given a finite number of functions $f_1, f_2, \dots, f_n$ belonging to $L^{r,\varphi}$, $1 \leqslant r < \infty$, and a number $\varepsilon > 0$, we can find step-functions $h_1, h_2, \dots, h_n$ such that $\mathfrak{M}_{r,\varphi}[f_i - h_i] < \varepsilon$ and that, for every sequence of constants $c_1, c_2, \dots, c_n$, we have $\mathfrak{M}_{k,\varphi}[h] \leqslant \mathfrak{M}_{k,\varphi}[f]$, where $f = c_1 f_1 + \dots + c_n f_n$, $h = c_1 h_1 + \dots + c_n h_n$, $1 \leqslant k \leqslant \infty$.*

If the intervals (a, b) and $(\varphi(a), \varphi(b))$ are both finite, this is immediate. For if h_i is a function of type s (see (2)) corresponding to f_i, and if the sub division $a_0, a_1, a_2, \ldots$ is sufficiently dense, then $\mathfrak{M}_{r,\varphi}[f_i - h_i] < \varepsilon$. If the subdivision is the same for all f_i, then h is a step-function of type s corresponding to f and so, in view of (3), $\mathfrak{M}_{k,\varphi}[h] \leqslant \mathfrak{M}_{k,\varphi}[f]$ for every $1 \leqslant k \leqslant \infty$ (if $\mathfrak{M}_{k,\varphi}[f] = \infty$, there is nothing to prove).

To prove (ii) in the general case, we proceed as in the last stage of the proof of (i). We write $f_i = f_i' + f_i''$, where f_i' is equal to f_i in an interval (a', b') and vanishes outside it, $f' = c_1 f_1' + c_2 f_2' + \ldots$, $f'' = c_1 f_1'' + c_2 f_2'' + \ldots$ Let h_i' be a function of type s corresponding to the function f_i in the interval (a', b'); outside (a', b') we put $h_i' = 0$. If $h = c_1 h_1' + c_2 h_2' + \ldots$, then we may suppose that h corresponds to the function f' in (a', b'), and so

$$\mathfrak{M}_{k,\varphi}[h; a, b] = \mathfrak{M}_{k,\varphi}[h; a', b'] \leqslant \mathfrak{M}_{k,\varphi}[f'; a', b'] \leqslant \mathfrak{M}_{k,\varphi}[f; a, b].$$

9.22. Let us fix two intervals $u \leqslant t \leqslant u_1$, $v \leqslant t \leqslant v_1$, and two non-decreasing functions $\varphi(t)$, $u \leqslant t \leqslant u_1$ and $\psi(t)$, $v \leqslant t \leqslant v_1$. We suppose that we have an operation T associating with every function $f(t)$, $u \leqslant t \leqslant u_1$, belonging to a class $\mathfrak{F}$, another function $g(t) = T[f]$, $v \leqslant t \leqslant v_1$. The functions f and g may even be undefined in sets of measure 0, the former with respect to φ, the latter with respect to ψ. As regards the class $\mathfrak{F}$, we suppose that, if $f_1 \in \mathfrak{F}$, $f_2 \in \mathfrak{F}$, and if c_1 and c_2 are arbitrary constants, then $c_1 f_1 + c_2 f_2 \in \mathfrak{F}$. The operation T is to be an *additive* operation, that is $T[c_1 f_1 + c_2 f_2] = c_1 T[f_1] + c_2 T[f_2]$ for any constants c_1, c_2.

T will be said to be of type (a, b) if $T[f]$ is defined for every $f \in L^{a,\varphi}(u, u_1)$, and if

$$(1) \qquad \mathfrak{M}_{b,\psi}[T[f]; v, v_1] \leqslant M\, \mathfrak{M}_{a,\varphi}[f; u, u_1],$$

where M is independent of f; in particular $T[f] \in L^{b,\psi}(v, v_1)$. The least value of M satisfying (1) will be called the *modulus* of the operation and denoted by $M_{\alpha\beta}$, where $\alpha = 1/a$, $\beta = 1/b$. The operation T is a linear operation in the sense of § 4.52.

It may happen that an operation T is defined not for all $f \in L^{a,\varphi}$ but only for a set S of f everywhere dense in $L^{a,\varphi}$ (the distance of two functions f_1 and f_2 being defined as $\mathfrak{M}_{a,\varphi}[f_1 - f_2]$), and that (1) is satisfied for all $f \in S$. Moreover suppose that S contains linear combinations of its elements. Then, without changing the values of $T[f]$ for $f \in S$, the operation T may be defined by continuity in the whole space $L^{a,\varphi}$ in such a way that it becomes of type (a, b) and that, moreover,

$$M_{\alpha\beta} = \operatorname{Sup} \mathfrak{M}_{b,\psi}[T[f]]/\mathfrak{M}_{a,\varphi}[f] \quad \text{for } f \in S.$$

For if $f \in L^{a,\varphi}$, $f_n \in S$, $n = 1, 2, \ldots$, $\mathfrak{M}_{a,\varphi}[f - f_n] \to 0$, then $\mathfrak{M}_{a,\varphi}[f_m - f_n] \to 0$ as $m, n \to \infty$, and hence, by (1), $\mathfrak{M}_{b,\psi}[T[f_m] - T[f_n]] \to 0$. From

Theorem 4.2, using the definition 9.21(1), we deduce that there is a function $g(t)$, which we may denote by $T[f]$, such that $\mathfrak{M}_{b,\psi}[T[f] - T[f_n]] \to 0$. The function $T[f]$ is defined outside a set of measure 0 with respect to ψ and is independent of the choice of $\{f_n\}$. If (1) is satisfied with f replaced by f_n, it holds for f also.

A particularly important case is the one in which S is the set S of § 9.21.

9.23. *Let T be an operation which is simultaneously of type (a_1, b_1) and of type (a_2, b_2), where $a_i = 1/\alpha_i$, $b_i = 1/\beta_i$, and the points $P_i = (\alpha_i, \beta_i)$ belong to the triangle (Δ) $0 \leqslant \alpha \leqslant 1$, $0 \leqslant \beta \leqslant \alpha$. Then T may be extended in such a way as to become of type (a, b) for every (α, β) on the segment l joining (α_1, β_1) and (α_2, β_2). Moreover the function $M_{\alpha\beta}$ is multiplicatively convex on l.*

Suppose that $\beta > 0$, i. e. that l does not lie on the α-axis.

Let $P = P(\alpha, \beta)$, $P = t_1 P_1 + t_2 P_2$, $t_i > 0$, $t_1 + t_1 = 1$. From what we have said it follows that it is enough to consider functions belonging to the set S of § 9.21. This set S is everywhere dense in every class $L^{a,\varphi}$, $1 \leqslant a < \infty$ [1]). If f belongs to S, then $f = x_1 f_1 + x_2 f_2 + \ldots + x_n f_n$, where f_i is the characteristic function of an interval over which the variation of φ is equal to ρ_i. If $g = T[f]$, $g_i = T[f_i]$, then $g = x_1 g_1 + \ldots + x_n g_n$. Since $f_i \in L^{a_1,\varphi}$, $f_i \in L^{a_2,\varphi}$, hence $g_i \in L^{b_1,\psi}$, $g_i \in L^{b_2,\psi}$; since b is contained between b_1 and b_2, we obtain, by Hölder's inequality, that $g_i \in L^{b,\psi}$. We can therefore find a step-function g_i^* such that we shall have $\mathfrak{M}_{b,\psi}[g_i - g_i^*] < \varepsilon$. Let $g^* = x_1 g_1^* + \ldots + x_n g_n^*$; we may also suppose that $\mathfrak{M}_{k,\psi}[g^*] \leqslant \mathfrak{M}_{k,\psi}[g]$, $1 \leqslant k \leqslant \infty$, for all values of $x_1, x_2, \ldots, x_n$ (§ 9.21(ii)).

Let ω be the maximum, with respect to the variables $x_1, x_2, \ldots, x_n$, of the ratio $(|x_1| + \ldots + |x_n|)/(\rho_1 |x_1|^a + \ldots + \rho_n |x_n|^a)^\alpha$ at the point P. Let $\eta = \omega\varepsilon$; since, by Minkowski's inequality, $|\mathfrak{M}_{b,\psi}[g] - \mathfrak{M}_{b,\psi}[g^*]|$ does not exceed $\varepsilon(|x_1| + \ldots + |x_n|)$ we see that

$$(1) \qquad \mathfrak{M}_{b,\psi}[g]/\mathfrak{M}_{a,\varphi}[f] \leqslant \eta + \mathfrak{M}_{b,\psi}[g^*]/(\Sigma \rho_k |x_k|^a)^\alpha.$$

Denoting by $X_1, X_2, \ldots, X_m$ certain linear forms of the variables $x_1, x_2, \ldots, x_n$, and by $\sigma_1, \sigma_2, \ldots, \sigma_m$ certain positive constants, we may represent the numerator of the last fraction in the form

---

[1]) This is not true if $a = \infty$.

$(\sigma_1 |X_1|^b + \dots + \sigma_m |X_m|^b)^\beta$. Using Theorem 9.2, we see that this fraction does not exceed

$$\operatorname{Sup}\left\{\frac{\mathfrak{M}_{b_1,\psi}[g^*]}{\mathfrak{M}_{a_1,\varphi}[f]}\right\}^{t_1} \operatorname{Sup}\left\{\frac{\mathfrak{M}_{b_2,\psi}[g^*]}{\mathfrak{M}_{a_2,\varphi}[f]}\right\}^{t_2} \leqslant \operatorname{Sup}\left\{\frac{\mathfrak{M}_{b_1,\psi}[g]}{\mathfrak{M}_{a_1,\varphi}[f]}\right\}^{t_1} \operatorname{Sup}\left\{\frac{\mathfrak{M}_{b_2,\psi}[g]}{\mathfrak{M}_{a_2,\varphi}[f]}\right\}^{t_2}$$

where the upper bounds are taken with respect to $x_1, \dots, x_n$. Thus the left-hand side of (1) does not exceed $\eta + M_{\alpha_1\beta_1}^{t_1} M_{\alpha_2\beta_2}^{t_2}$ and, η being arbitrarily small with ε, we obtain

$$(2) \qquad M_{\alpha\beta} \leqslant M_{\alpha_1\beta_1}^{t_1} M_{\alpha_2\beta_2}^{t_2}.$$

From this we deduce the first part of the theorem. Since (α_1, β_1) and (α_2, β_2) may be any pair of points on the segment l, the inequality (2) proves the second part of the theorem also.

It remains to prove the theorem in the case of l lying on the α-axis. This case has no interesting application and we consider it for the sake of completness only. Suppose first that the number $l = \psi(v_1) - \psi(v)$ is finite. If the operation T is of type (a_1, ∞) and of type (a_2, ∞), where $0 \leqslant 1/a_1 = \alpha_1 < \alpha_2 = 1/a_2$, then T is also of type $(a_2, 1/\eta)$ for every $\eta > 0$. Since the expression $\left(\frac{1}{l}\int_v^{v_1} |g|^{1/\eta} d\psi\right)^\eta$ increases as η decreases to 0 (§ 4.15) and tends to the essential upper bound of g (with respect to the function ψ), we deduce that $M_{\alpha_2\eta} \leqslant l^\eta M_{\alpha_2 0}$. Hence, if $f \in S$, $g = T[f]$, and if (α, η_1), $\alpha = t_1\alpha_1 + t_2\alpha_2$, $t_1 + t_2 = 1$, is a point on the segment joining $(\alpha_1, 0)$ and (α_2, η), then

$$\left(\int_v^{v_1} |g|^{1/\eta_1} d\psi\right)^{\eta_1} \leqslant M_{\alpha_1 0} (l^\eta M_{\alpha_2 0})^{t_2} \left(\int_u^{u_1} |f|^{1/\alpha} d\varphi\right)^\alpha$$

and, making $\eta \to 0$, we obtain $M_{\alpha 0} \leqslant M_{\alpha_1 0}^{t_1} M_{\alpha_2 0}^{t_2}$.

To remove the condition $l < \infty$, let (v', v_1') be an interval interior to (v, v_1). Considering the function g in (v', v_1') only, we have a linear operation with norm $M'_{\alpha 0} \leqslant M_{\alpha 0}$. We have $M'_{\alpha 0} \leqslant M'^{t_1}_{\alpha_1 0} M'^{t_2}_{\alpha_2 0} \leqslant M^{t_1}_{\alpha_1 0} M^{t_2}_{\alpha_2 0}$ and, making $v' \to v$, $v_1' \to v_1$, we obtain $M_{\alpha 0} \leqslant M^{t_1}_{\alpha_1 0} M^{t_2}_{\alpha_2 0}$.

9.24. It is natural to inquire how far the condition imposed upon the point (α, β) to remain within the triangle Δ is essential for the truth of the theorem. The results are mostly negative. For details we refer the reader to M. Riesz [3].

Having in view definite applications we supposed in Theorem 9.2 that the coefficients of the linear forms X_j, as well as the variables x_k, were complex numbers. Similarly in Theorem 9.23 the functions f and $T[f]$ were complex functions of a real variable. In some cases however it is important to have those theorems for real variables. Theorem 9.2 holds, and its proof is unaffected, if we assume that the numbers a_{jk}, x_k are real. Similarly Theorem 9.23, which follows from Theorem 9.2 by passages to limits, remains true in the domain of real variables.

9.25. As an application of Theorem 9.23 we shall prove the following theorem, stated without proof in § 4.63. *If $r < s < r'$, the class (L^r, L^r) is contained in (L^s, L^s).* Consider the series

$$(1)\ \frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx),\qquad (1a)\ \frac{1}{2}a_0\lambda_0+\sum_{n=1}^{\infty}\lambda_n(a_n\cos nx+b_n\sin nx),$$

and suppose that, whenever (1) is the Fourier series of a function $f\in L^r$, (1a) is the Fourier series of a function $g=T[f]\in L^r$. We shall prove first that $g=T[f]$ is an operation of type (r,r) in the sense of § 9.22. It is plain that $T[f]$ is an additive operation and it remains to prove the existence of a constant M such that $\mathfrak{M}_r[g]\leqslant M\mathfrak{M}_r[f]$. Let $\sigma_n^*(x)$ and $l_n(x)$ denote the $(C,1)$ means of the series (1a) and of the series $\frac{1}{2}\lambda_0+\lambda_1\cos x+\ldots$ respectively. From the formula

$$(2)\qquad \sigma_n^*(x)=\frac{1}{\pi}\int_0^{2\pi}f(x+t)\,l_n(t)\,dt$$

(§ 4.64), we obtain $|\sigma_n^*(x)|\leqslant\pi^{-1}\mathfrak{M}_r[f]\,\mathfrak{M}_{r'}[l_n]$, $\mathfrak{M}_r[\sigma_n^*]\leqslant 2\mathfrak{M}_r[f]\,\mathfrak{M}_{r'}[l_n]$, so that, for fixed n, (2) is an operation associating with every $f\in L^r$ a function $\sigma_n^*\in L^r$. Let M_n be the modulus of the operation (2). Since, by hypothesis, $\mathfrak{M}_r[\sigma_n^*]$ is bounded for every $f\in L^r$, the sequence $\{M_n\}$ is bounded (§ 4.55). If $M=\operatorname{Sup}M_n$, we have $\mathfrak{M}_r[\sigma_n^*]\leqslant M\mathfrak{M}_r[f]$ and, making $n\to\infty$, $\mathfrak{M}_r[g]\leqslant M\mathfrak{M}_r[f]$. This shows that $T[f]$ is of type (r,r).

Now it is easy to complete the proof. In view of Theorem 4.63(ii), T is also of type (r',r'), and from Theorem 9.23 we see that T is of type (s,s), where s is any number such that $1/s$ is contained between $1/r$ and $1/r'$.

9.3. Proof of F. Riesz's theorem. To prove Theorem 9.11(a) let

$$(1)\qquad c_n=\int_a^b f(t)\overline{\varphi_n(t)}\,dt,\qquad n=1,2,\ldots$$

be the n-th Fourier coefficient of f (§ 1.31). Then

$$(2)\qquad \sum_{n=1}^{\infty}|c_n|^2 \leqslant \int_a^b |f(t)|^2\,dt, \qquad |c_n| \leqslant M\int_a^b |f(t)|\,dt,$$

where the first inequality is Bessel's inequality and the second follows from (1) and the inequality $|\varphi_n| \leqslant M$. Let us put $\varphi(x)=x$, and let $\psi(x)=[x]$ for $x>0$, $\psi(x)=0$ elsewhere. If $c(x)$ is equal to c_n for $x=n$ and is arbitrary elsewhere, the inequalities (2) may be written

$$\mathfrak{M}_{2,\psi}[c] \leqslant \mathfrak{M}_{2,\varphi}[f], \quad \mathfrak{M}_{\infty,\psi}[c] \leqslant M\,\mathfrak{M}_{1,\varphi}[f],$$

so that the operation $c(x)=T[f]$ is of types $(2,2)$ and $(1,\infty)$. In view of Theorem 9.23, T is also of type (p,p'), where $p=1/\alpha$, $p'=1/(1-\alpha)$, $\frac{1}{2}\leqslant\alpha\leqslant 1$. Since $M_{1/2,1/2}\leqslant 1$, $M_{1,0}\leqslant M$, we find, using Theorem 9.23 again, that

$$M_{\alpha,1-\alpha} \leqslant M_{1,0}^{(\alpha-1/2)/(1-1/2)}\, M_{1/2,1/2}^{(1-\alpha)/(1-1/2)} \leqslant M^{2\alpha-1} = M^{(2-p)/p}.$$

Hence $\mathfrak{M}_{p',\psi}[c] \leqslant M^{(2-p)/p}\,\mathfrak{M}_{p,\varphi}[c]$, and this is just Theorem 9.11(a).

To prove Theorem 9.11(b) we argue similarly, starting from the inequalities

$$\int_a^b |f(t)|^2\,dt \leqslant \sum_{n=1}^{\infty}|c_n|^2, \quad |f(t)| \leqslant M\sum_{n=1}^{\infty}|c_n|,$$

where f is the function the existence of which is assured by the Riesz-Fischer theorem (§ 4.21(1)). The details may be left to the reader.

9.31. We complete the above proof by a few general remarks. In the first place we observe that the apparatus of the Stieltjes-Lebesgue integral was not really necessary in the proof of Theorem 9.11(a). For, if we put $c(x)=c_n$ for $n-1\leqslant x<n$, $n=1,2,\ldots$, the inequalities 9.3(2) may be writen $\mathfrak{M}_2[c]\leqslant\mathfrak{M}_2[f]$, $\mathfrak{M}_\infty[c]\leqslant M\,\mathfrak{M}[f]$, where the integrals are ordinary Lebesgue integrals, and we may apply Theorem 9.23 in the case $\varphi(x)=x$, $\psi(x)=x$. This course is slightly less simple in the case of Theorem 9.11(b); but, as we know, both parts of Theorem 9.11 can easily be deduced from each other (see also § 9.9.1).

The proof of F. Riesz's Theorem can be made more elementary by basing it on Theorem 9.2 instead of Theorem 9.23. But the application of the latter theorem has two advantages. The first of them is that it clearly shows the proper place of Theorem 9.11, which turns out to be not a generalization but a consequence of the Riesz-Fischer theorem. Besides, Theorem 9.23 is of fundamental character and may be applied, so to speak, automatically in many cases where an application of Theorem 9.2 would require certain calculations, which would amount substantially to a proof of Theorem 9.23.

We also observe that in § 9.3 we applied the Bessel inequality and the Riesz-Fischer theorem for a complex system $\{\varphi_n\}$, whereas the proofs given in §§ 1.6, 4.21 bear on the case of real φ_n. The reader will have no difficulty in adapting those proofs to the case of complex φ_n.

9.4. Theorems of Paley. The Hausdorff-Young theorems are not the only results which connect the type of integrability of a function with the exponent of convergence of its coefficients. Further results in this direction have been obtained by Hardy and Littlewood. The simplest way to them seems to lead through theorems of Paley which partly generalize the Hardy-Littlewood theorems and bear on general orthogonal and normal systems of uniformly bounded functions.

Given any sequence of complex numbers $c_1, c_2, \ldots$ tending to 0, we denote by $c_1^*, c_2^*, \ldots$ the sequence $|c_1|, |c_2|, \ldots$ rearranged in descending order of magnitude. If several $|c_n|$ are equal, then there are corresponding repetitions in the c_n^*. We put

$$\left\{\sum_{n=1}^{\infty}|c_n|^r n^{r-2}\right\}^{1/r} = \mathfrak{B}_r[c], \qquad \left\{\sum_{n=1}^{\infty} c_n^{*r} n^{r-2}\right\}^{1/r} = \mathfrak{B}_r[c^*].$$

Let $\varphi_1(x), \varphi_2(x), \ldots$ be a system of functions which are orthogonal, normal, and uniformly bounded ($|\varphi_n| \leqslant M$, $n = 1, 2, \ldots$) in an interval (a, b). Writing $\mathfrak{M}_r[f] = \mathfrak{M}_r[f; a, b]$, Paley's theorems may be stated as follows [1]).

(i) *If, for a sequence of numbers $c_1, c_2, \ldots$, the expression $\mathfrak{B}_q[c^*]$ is finite, there is a function $f \in L^q$ such that c_n is the Fourier coefficient of f with respect to φ_n, $n = 1, 2, \ldots$, and*

$$\mathfrak{M}_q[f] \leqslant A_q \mathfrak{B}_q[c^*], \tag{1}$$

where A_q depends only on q and M.

---

[1]) Paley [4].

(ii) *If $f \in L^p$, and if $c_1, c_2, \ldots$ are the Fourier coefficients of f with respect to $\{\varphi_n\}$, then $\mathfrak{B}_p^*[c] < \infty$ and*

$$\mathfrak{B}_p^*[c] \leqslant A'_p \mathfrak{M}_p[f], \tag{2}$$

where A'_p depends only on p and M.

The reason why we introduced the starred sequence $\{c_n^*\}$ becomes clear from the following considerations. Let $a_1, a_2, \ldots, b_1, b_2, \ldots$ be two sequences of non-negative numbers, and let S be the sum $a_1 b_1 + a_2 b_2 + \ldots$; S may also be infinite. We suppose that $\{a_n\}$ is either non-increasing or non-decreasing. *Rearranging $\{b_n\}$ in all possible manners, we obtain for S the largest value when $\{a_n\}$ and $\{b_n\}$ vary in the same sense,* i. e. if they are either both non-increasing or both non-decreasing; *S is a minimum when $\{a_n\}$ and $\{b_n\}$ vary in opposite senses.* To fix ideas we assume that $a_1 \geqslant a_2 \geqslant \ldots$ To prove the first part of the proposition we observe that, if e. g. $a_1 > a_2$ and $b_1 < b_2$, then, replacing $a_1 b_1 + a_2 b_2$ by $a_1 b_2 + a_2 b_1$, we increase S by $(a_1 - a_2)(b_2 - b_1) > 0$. Similarly we prove the second part.

Hence, considering all possible rearrangements of $\{|c_n|\}$, we see that $\mathfrak{B}_q[c]$ is a minimum when $\{|c_n|\}$ is arranged in descending order of magnitude. With this arrangement the expression $\mathfrak{B}_p[c]$ attains its maximum. It follows that, if (1) and (2) are true, the inequalities which we obtain by replacing $\mathfrak{B}_q^*[c], \mathfrak{B}_p^*[c]$ by $\mathfrak{B}_q[c], \mathfrak{B}_q[c]$, hold à fortiori. On the other hand, since the order of the functions φ_n within the sequence $\{\varphi_n\}$ is irrelevant, we may change this order, if necessary, and suppose from the very beginning that $c_n^* = |c_n|$. It is therefore sufficient to prove (1) and (2) with c_n^* replaced by $|c_n|$, and in the subsequent proof we shall write $|c_n|$ instead of c_n^*.

9.401. Since, by Hölder's inequality,

$$\Sigma |c_n|^2 = \Sigma |c_n|^2 n^{2(q-2)/q} n^{-2(q-2)/q} \leqslant (\Sigma |c_n|^q n^{q-2})^{2/q} (\Sigma n^{-2})^{(q-2)/q},$$

we see that, under the hypothesis of Theorem 9.4(i), the numbers c_n are the Fourier coefficients, with respect to $\{\varphi_n\}$, of a function $f(x) \in L^2$. Let $s_n(x)$ be the n-th partial sum of the series $c_1 \varphi_1(x) + c_2 \varphi_2(x) + \ldots$ It is sufficient to prove that $\mathfrak{M}_q[s_{2^N-1}] \leqslant A_q \mathfrak{B}_q[c]$, $N = 1, 2, \ldots$, for, since $\mathfrak{M}_2[f - s_{2^{N_i}-1}] \to 0$, there is a sequence of integers $\{N_i\}$ such that $s_{2^{N_i}-1}(x)$ converges almost everywhere to

$f(x)$ (§ 4.2), and an application of Fatou's lemma to the last inequality gives 9.4(1).

Let $C_\mu = \sum_{m=2^{\mu-1}}^{2^\mu-1} |c_m|^q m^{q-2}$, $\Phi_\mu = \sum_{m=2^{\mu-1}}^{2^\mu-1} c_m \varphi_m$, $\mu = 1, 2, \dots$, and let $\nu \geqslant \mu$. We begin by proving that

$$\text{(1)} \qquad \int_a^b |\Phi_\mu \Phi_\nu|^{q/2} dx \leqslant B_q\, C_\mu^{\frac12} C_\nu^{\frac12}\, 2^{-\frac14(\nu-\mu)}, \quad q \geqslant 4,$$

where B_q is independent of $\{c_n\}$. For, since $|\varphi_m| \leqslant M$, the left-hand side of (1) does not exceed

$$\operatorname{Max} \{|\Phi_\mu|^{\frac12 q} |\Phi_\nu|^{\frac12 q-2}\} \int_a^b |\Phi_\nu|^2 dx \leqslant$$

$$\leqslant M^{q-2} \Big(\sum_{m=2^{\mu-1}}^{2^\mu-1} |c_m|\Big)^{\frac12 q} \Big(\sum_{n=2^{\nu-1}}^{2^\nu-1} |c_n|\Big)^{\frac12 q-2} \Big(\sum_{p=2^{\nu-1}}^{2^\nu-1} |c_p|^2\Big).$$

Writing $|c_m| = |c_m| m^{(q-2)/q} \cdot m^{-(q-2)/q}$, $|c_n| = |c_n| n^{(q-2)/q} \cdot n^{-(q-2)/q}$, $|c_p|^2 = |c_p|^2 p^{2(q-2)/q} \cdot p^{-2(q-2)/q}$, applying Hölder's inequalities so as to introduce the sums C_μ, C_ν, and observing that

$$\sum_{m=2^{\mu-1}}^{2^\mu-1} m^{-\alpha} < \int_0^{2^\mu} x^{-\alpha} dx, \ \alpha > 0; \qquad \sum_{p=2^{\nu-1}}^{2^\nu-1} p^{-2} < \frac{1}{2^{2(\nu-1)}} + \int_{2^{\nu-1}}^{\infty} \frac{dx}{x^2} < 4.2^{-\nu},$$

we easily obtain the inequality (1) with B_q not exceeding

$$M^{q-2}\, 4^{(q-2)/q} (q-1)^{\frac12(q-1)} (q-1)^{(q-1)(q-4)/2q} < M^{q-2} q^q.$$

Now, supposing that $q \geqslant 4$ is an integer, we have

$$\mathfrak{M}_q^q[s_{2^N-1}] = \int_a^b \Big|\sum_{\nu=1}^N \Phi_\nu\Big|^q dx \leqslant \sum_{\nu_1=1}^N \sum_{\nu_2=1}^N \dots \sum_{\nu_q=1}^N \int_a^b |\Phi_{\nu_1} \dots \Phi_{\nu_q}|\, dx.$$

Writing $|\Phi_{\nu_1}\Phi_{\nu_2}\dots\Phi_{\nu_q}| = |(\Phi_{\nu_1}\Phi_{\nu_2})(\Phi_{\nu_1}\Phi_{\nu_3})\dots(\Phi_{\nu_1}\Phi_{\nu_q})\dots(\Phi_{\nu_{q-1}}\Phi_{\nu_q})|^{1/(q-1)}$, where the number of bracketed factors is $Q = \frac12 q(q-1)$, and applying Hölder's inequality with the exponents Q (§ 4.141), we obtain

$$\int_a^b |\Phi_{\nu_1}\Phi_{\nu_2}\dots\Phi_{\nu_q}|\, dx \leqslant \prod_{\substack{i,j=1\\ i<j}}^q \Big\{\int_a^b |\Phi_{\nu_i}\Phi_{\nu_j}|^{\frac12 q} dx\Big\}^{1/Q} \leqslant$$

$$\leqslant B_q \prod_{\substack{i,j=1\\ i<j}}^q C_{\nu_i}^{1/2Q} C_{\nu_j}^{1/2Q}\, 2^{-|\nu_i-\nu_j|/2Q} = B_q \prod_{i=1}^q C_{\nu_i}^{1/q} \Big\{\prod_{j=1}^{q}{}^{(i)}\, 2^{-|\nu_i-\nu_j|/4Q}\Big\}.$$

Here the upper suffix (i) indicates that the factor $j = i$ (which, by the way, is equal to 1) is omitted. Substituting this in the right-hand side of the inequality for $\mathfrak{M}_q^q[s_2{}^N{}_{-1}]$, and applying Hölder's inequality with the exponents q, we obtain

$$\mathfrak{M}_q^q[s_2{}^N{}_{-1}] \leqslant B_q \prod_{i=1}^{q} \left\{ \sum_{\nu_1=1}^{N} \dots \sum_{\nu_q=1}^{N} C_{\nu_i} \prod_{j=1}^{q\,(i)} 2^{-|\nu_i - \nu_j|/2(q-1)} \right\}^{1/q}.$$

Consider the multiple sum in curly brackets. Summing first with respect to $\nu_1, \nu_2, \dots, \nu_{i-1}, \nu_{i+1}, \dots, \nu_q$, and then with respect to ν_i, we obtain that the sum considered does not exceed

$$\left(\sum_{\nu=1}^{N} C_\nu\right) \left\{ \sum_{\nu=-\infty}^{+\infty} 2^{-|\nu|/2(q-1)} \right\}^{q-1}, \text{ and so}$$

$$\mathfrak{M}_q^q[s_2{}^N{}_{-1}] \leqslant B_q \left\{ \sum_{\nu=1}^{\infty} C_\nu \right\} \left\{ \sum_{\nu=-\infty}^{\infty} 2^{-|\nu|/2(q-1)} \right\}^{q-1} = A_q^q\, \mathfrak{B}_q^q[c],$$

where $A_q^q = B_q\{\Sigma\, 2^{-|\nu|/2(q-1)}\}^{q-1}$. Thus the theorem is proved for $q = 4, 5, \dots$; and it is plainly true also for $q = 2$.

To prove the theorem in the general case we observe that the inequality 9.4(1) may be written

$$\mathfrak{M}_q^q[f] \leqslant A_q^q \sum_{n=1}^{\infty} (|c_n|\, n)^q\, n^{-2},$$

and that $f(t) = \Sigma\, nc_n \cdot \varphi_n(t)/n$ is obtained by a linear transformation from the numbers nc_n. Thus, arguing as in § 9.3, we may interpolate by means of Theorem 9.23, and Theorem 9.4(i) is established completely.

9.402. Theorem 9.4(ii) may be obtained by an argument similar to that of § 9.121. We put $p' = q$, fix an integer $N > 0$, and denote by $g(x)$ a sum $d_1 \varphi_1(x) + d_2 \varphi_2(x) + \dots + d_N \varphi_N(x)$, where the numbers $d_1, d_2, \dots, d_N$ will be defined in a moment. Then

$$(1) \qquad \int_a^b f\bar{g}\, dx = \sum_{n=1}^{N} c_n \bar{d}_n = \sum_{n=1}^{N} c_n\, n^{(p-2)/p} \cdot \bar{d}_n\, n^{(q-2)/q}.$$

Let us apply Hölder's inequality, with the exponents p and q, to the last sum. If sign d_n = sign c_n, $|c_n|^p n^{p-2} = |d_n|^q n^{q-2}$, the inequality degenerates into equality (§ 4.12); hence, applying Hölder's inequality to the integral in (1), we obtain

$$\left(\sum_{n=1}^{N} |c_n|^p\, n^{p-2}\right)^{1/p} \left(\sum_{n=1}^{N} |d_n|^q\, n^{q-2}\right)^{1/q} \leqslant \mathfrak{M}_p[f]\, \mathfrak{M}_q[g].$$

In virtue of 9.4(1), the second factor on the right does not exceed $A_q \mathfrak{B}_q[d]$, so that

$$\sum_{n=1}^{N} |c_n|^p n^{p-2} \leqslant A_q^p \int_a^b |f|^p dx.$$

Making $N \to \infty$ we obtain the inequality 9.4(2) with $A'_p = A_q$.

The reader will easily convince himself that $A_q \leqslant M^{(q-2)/q} \alpha_q$, and so $A'_p \leqslant M^{(2-p)/p} \alpha'_p$, where α_q depends only on q, and α'_p only on p.

9.41. It is an interesting fact that Paley's theorems contain the theorems of F. Riesz as special cases, although in a slightly less precise form: into the right-hand sides of the inequalities $\mathfrak{M}_{p'}[f] \leqslant M^{(2-p)/p} \mathfrak{N}_p[c]$, $\mathfrak{N}_{p'}[c] \leqslant M^{(2-p)/p} \mathfrak{M}_p[f]$ we shall have to introduce a numerical factor β_p depending on p. In view of the last remark of 9.402 it is sufficient to show that

(1) $$\mathfrak{B}_q[c^*] \leqslant \gamma_q \mathfrak{N}_{q'}[c], \quad \mathfrak{B}_p[c^*] \geqslant \gamma'_p \mathfrak{N}_{p'}[c],$$

where γ_q depends only on q, and γ'_p only on p. We shall prove the first of these inequalities only; the proof of the second is similar.

First of all we observe that, if $x, y, \ldots$ are non-negative numbers, then $(x+y+\ldots)^r \leqslant x^r + y^r + \ldots$ for $0 < r \leqslant 1$, and $(x+y+\ldots)^r \geqslant x^r + y^r + \ldots$ for $r \geqslant 1$. The first of these inequalities has already been established in the case of two terms (§ 4.13), and in the general case the proof follows by induction; the second inequality may be obtained in the same way. Now

$$\sum_{n=1}^{\infty} c_n^{*q} n^{q-2} = \sum_{\nu=0}^{\infty} \sum_{n=2^\nu}^{2^{\nu+1}-1} c_n^{*q} n^{q-2} \leqslant$$

$$\leqslant 2^{q-2} \sum_{\nu=0}^{\infty} c_{2^\nu}^{*q} 2^{\nu(q-1)} = 2^{q-2} \sum_{\nu=0}^{\infty} (c_{2^\nu}^{*q'} 2^\nu)^{q-1} \leqslant$$

$$\leqslant 2^{q-2} \left(\sum_{\nu=0}^{\infty} c_{2^\nu}^{*q'} 2^\nu \right)^{q-1} \leqslant 2^{2q-3} \left(c_1^* + \sum_{\nu=1}^{\infty} c_{2^\nu}^{*q'} 2^{\nu-1} \right)^{q-1} \leqslant$$

$$\leqslant 2^{2q-3} \left(c_1^* + \sum_{\nu=1}^{\infty} \sum_{n=2^{\nu-1}}^{2^\nu - 1} c_n^{*q'} \right)^{q-1} \leqslant 2^{2q-3} \left(2 \sum_{n=1}^{\infty} c_n^{*q'} \right)^{q-1} = 2^{3q-4} \mathfrak{N}_{q'}^q[c],$$

and the first inequality (1) is established.

This result might suggest that, perhaps, the theorems of Paley and those of F. Riesz are, roughly speaking, equivalent. But this is not so. For if we put e. g. $\varphi_n(x) = \cos nx$, $(a, b) = (0, \pi)$, $c_n = n^{-3/4} \log^{-3/4}(n+1)$, $n = 1, 2, \dots$, then $\mathfrak{B}_4[c] < \infty$, and so, by Theorem 9.4(i), the function $f(x) = c_1 \cos x + c_2 \cos 2x + \dots$ belongs to L^4. Since $\mathfrak{N}_{4/3}[c] = \infty$, this result cannot be obtained from Theorem 9.11(b).

9.42. Given a real function $f(x)$, $a \leqslant x \leqslant b$, we shall denote by $\mathrm{E}(f > y)$ the set of points where $f(x) > y$. The functions f and φ will be called *equimeasurable* functions if $|\mathrm{E}(f > y)| = |\mathrm{E}(\varphi > y)|$ for every y[1]. Each of these functions may be thought of as obtained from the other by a sort of 'rearrangement' of the argument x, although we should find some difficulty if we tried to define this rearrangement precisely. It is plain that if one of two functions equimeasurable in the interval (a, b) is integrable, so is the other and their integrals over (a, b) are equal.

For every measurable function $f(x)$ defined in a finite interval $a \leqslant x \leqslant b$, there is a function $f^*(x)$, $a \leqslant x \leqslant b$, equimeasurable with $f(x)$ and non-increasing. For let $m(y) = |\mathrm{E}(f > y)|$ and suppose for simplicity that $a = 0$; then $f^*(x)$ may be defined as the function inverse to $m(y)$. The function f^* is defined uniquely except at its points of discontinuity. To fix ideas we may suppose that $f^*(x+0) = f^*(x)$. Similarly there is a function $f_*(x)$ equimeasurable with $f(x)$ and non-decreasing.

We shall require the following lemma.

If $f(x)$ is non-negative, then, for any function $g(x)$ which is non-negative and non-increasing, we have

$$(1) \qquad \int_a^b g f_* \, dx \leqslant \int_a^b g f \, dx \leqslant \int_a^b g f^* \, dx.$$

First of all we observe that, if $f_n(x)$ tends almost everywhere to $f(x)$, then $f_n^*(x) \to f^*(x)$, $f_{*n}(x) \to f_*(x)$ except at a set of points which is at most enumerable. This follows from the fact that, for every y, $|\mathrm{E}(f_n > y)| \to |\mathrm{E}(f > y)|$. Secondly, if $\{f_n\}$ is monotonic and tends to a limit $f(x)$, and if (1) is true for f_n, $n = 1, 2, \dots$, it is true also for f. This follows from the preceding remark and from Lebesgue's theorem on the integration of monotonic

---

[1]) This notion has been introduced by F. Riesz [8].

sequences. Now (1) is certainly true if (a, b) can be broken up into a number of intervals of equal length in each of which f, and so also f^* and f_*, is constant, for then the integrals (1) reduce to sums (§ 9.4). Since, starting with such functions, we may, by monotonic passages to limits, obtain any measurable function $f(x)$ [1]) (more precisely, a function equivalent to $f(x)$) the inequalities (1) are true in the general case.

9.43. Now we shall show that, if we invert the rôles of $f(x)$ and $\{c_n\}$ in Theorems 9.4, we obtain theorems which are equally true. It will simplify the proofs slightly if we suppose that the interval (a, b) is finite, but the proofs in the general case undergo but little change. We suppose for simplicity that (a, b) is of the form $(0, h)$. By f^* we shall denote the function which is non-increasing and equimeasurable with $|f|$, and write

$$\mathfrak{U}_r[f] = \left\{\int_0^h |f|^r x^{r-2}\, dx\right\}^{1/r}, \qquad \mathfrak{U}_r[f^*] = \left\{\int_0^h f^{*r} x^{r-2}\, dx\right\}^{1/r}\ ^{2)}.$$

If the functions φ_n satisfy the same conditions as before, then

(i) *If $\mathfrak{U}_q[f^*]$ is finite and if c_n is the Fourier coefficient of f with respect to φ_n, then $\mathfrak{N}_q[c]$ is finite and*

$$\mathfrak{N}_q[c] \leqslant A_q\, \mathfrak{U}_q[f^*], \tag{1}$$

where A_q depends only on q and M.

(ii) *If, for a sequence $\{c_n\}$, we have $\mathfrak{N}_p[c] < \infty$, the numbers c_n are the Fourier coefficients of a function f such that*

$$\mathfrak{U}_p[f^*] \leqslant A'_p\, \mathfrak{N}_p[c], \tag{2}$$

where $A'_p = A_{p'}$.

Since the proofs follow the same lines as those of Theorems 9.4, we shall condense some parts. We begin by proving (1) in a weaker form, with f^* replaced by $|f|$ on the right.

---

[1]) See e. g. Hobson, *Theory of functions*, 2, 376.

[2]) In the case $(a, b) = (-\infty, +\infty)$ it is convenient to define f^* as a function which is equimeasurable with $|f|$, even, and non-increasing in $(0, \infty)$, and to put $\mathfrak{U}_r^*[f] = \left\{\int_{-\infty}^{+\infty} f^{*r}\, |x|^{r-2}\, dx\right\}^{1/r}$.

The inequality is true for $q=2$, and so, if we prove it for $q=4, 5, \dots$, an application of Theorem 9.23 yields the result for general q. Let $f_\nu(x)$ be the function equal to $f(x)$ in the interval $(h2^{-\nu}, h2^{-\nu+1})$ and to 0 elsewhere, $\nu=1,2,\dots$, and let c_n^ν be the Fourier coefficient of f_ν with respect to φ_n, so that $c_n = c_n^1 + c_n^2 + \dots$ We fix an integer $N>0$ and observe that

$$\sum_{n=1}^{N} |c_n|^q = \sum_{n=1}^{N} |c_n^1 + c_n^2 + \dots|^q \leqslant \sum_{\nu_1=1}^{\infty} \dots \sum_{\nu_q=1}^{\infty} \left\{ \sum_{n=1}^{N} |c_n^{\nu_1} \dots c_n^{\nu_q}| \right\},$$

and that

$$\sum_{n=1}^{N} |c_n^{\nu_1} \dots c_n^{\nu_q}| \leqslant \prod_{\substack{i,j=1\\ i<j}}^{q} \left\{ \sum_{n=1}^{N} |c_n^{\nu_i} c_n^{\nu_j}|^{q/2} \right\}^{1/Q},$$

where $Q=\frac{1}{2}q(q-1)$. Now we prove that

$$\sum_{n=1}^{N} |c_n^{\mu} c_n^{\nu}|^{q/2} \leqslant B_q \eta_\mu^{1/2} \eta_\nu^{1/2} 2^{-1/2|\mu-\nu|}, \tag{3}$$

where $B_q \leqslant M^{q-2}\beta_q$ with β_q depending only on q, and η_ν equal to $\mathfrak{M}[|f|^q x^{q-2}; h2^{-\nu}, h2^{-\nu+1}]$. For the left-hand side of (3) is equal to

$$\sum_{n=1}^{N} \left| \int_{h2^{-\mu}}^{h2^{-\mu+1}} f\bar{\varphi}_n\,dx \right|^{1/2 q} \left| \int_{h2^{-\nu}}^{h2^{-\nu+1}} f\bar{\varphi}_n\,dx \right|^{1/2 q} \leqslant$$

$$\leqslant M^{q-2} \left(\int_{h2^{-\mu}}^{h2^{-\mu+1}} |f|\,dx \right)^{1/2 q} \left(\int_{h2^{-\nu}}^{h2^{-\nu+1}} |f|\,dx \right)^{1/2 q-2} \sum_{n=1}^{N} \left| \int_{h2^{-\nu}}^{h2^{-\nu+1}} f\bar{\varphi}_n\,dx \right|^2,$$

and, by Bessel's inequality, the last factor on the right does not exceed $\mathfrak{M}_2^2[f; h2^{-\nu}, h2^{-\nu+1}]$. Writing $|f| = |f|\,x^{(q-2)/q} x^{-(q-2)/q}$, $|f|^2 = |f|^2 x^{2(q-2)/q} x^{-2(q-2)/q}$, and applying Hölder's inequalities, we obtain (3). Hence, arguing as in § 9.401, we obtain the inequality $\left(\sum_{n=1}^{N} |c_n|^q\right)^{1/q} \leqslant A_q \mathfrak{U}_q[f]$, and (1) follows on making N tend to ∞.

So far we have proved (1) with f^* replaced by $|f|$. To obtain the exact inequality (1) let us assume first that f is a step-function. Rearranging the order of the intervals where f is constant, which amounts to an one-to-one transformation of the interval $(0, h)$ into itself, we transform $|f|$ into f^*. At the same time $f(x)$ is transformed into a function $h(x)$, and the functions φ_n are transformed into functions ψ_n, which again form an orthogonal and normal system. Since the Fourier coefficient of f with respect to φ_n is equal to that of h with respect to ψ_n, (1) follows, in our case, from the weaker inequality previously established.

To prove (1) in the general case, let $\{f_k\}$ be a sequence of functions for each of which (1) is true, so that

$$\sum_{n=1}^{N} |c_n^k|^q \leqslant A_q^q \int_0^h f_k^{*q}\, x^{q-2}\, dx, \tag{4}$$

where $N > 0$ is fixed, f_k^* is non-increasing and equimeasurable with $|f_k|$, and $c_1^k, c_2^k, \ldots$ are the Fourier coefficients of f_k. Since any bounded f is the limit of a uniformly bounded and almost everywhere convergent sequence $\{f_k\}$ of step-functions, and since $c_n^k \to c_n$, $f_k^*(x) \to f^*(x)$ as $k \to \infty$, we may replace c_n^k, f_k by c_n, f in (4). If f is arbitrary, we put $f_k(x) = f(x)$ if $|f(x)| \leqslant k$ and $f_k(x) = 0$ if $|f(x)| > k$. Hence again $c_n^k \to c_n$, $f_k^*(x) \leqslant f_{k+1}^*(x) \to f^*(x)$, and, since the f_k are bounded, (4) is true for f. The inequality (1) follows on making $N \to \infty$.

To prove (2) let us fix $N > 0$ and put $f_N = c_1\varphi_1 + \ldots + c_N\varphi_N$. We verify that

$$\mathfrak{U}_p[f_N^*] = \mathrm{Sup} \int_0^h f_N^*\, g\, dx \quad \text{for all } g \geqslant 0 \text{ with } \mathfrak{U}_{p'}[g] \leqslant 1.$$

It is even sufficient to restrict g to the domain of step-functions. A moment's consideration shows that, then, $\int_0^h f_N^* g\, dx = \int_0^N \bar{f}_N \gamma\, dx$, where the absolute value of the function $\gamma(x) = \gamma(x; g, N)$ is equimeasurable with g. Denoting the Fourier coefficients of γ by d_n, we have

$$\mathfrak{U}_p[f_N^*] = \mathrm{Sup}_g \int_0^h \bar{f}_N \gamma\, dx = \mathrm{Sup}_g \left| \sum_{n=1}^{N} c_n d_n \right| \leqslant \tag{5}$$

$$\leqslant \mathrm{Sup}_g \left(\sum_{n=1}^{N} |c_n|^p\right)^{1/p} \left(\sum_{n=1}^{N} |d_n|^{p'}\right)^{1/p'} \leqslant \mathrm{Sup}_g\, \mathfrak{N}_p[c]\, A_{p'}\, \mathfrak{U}_{p'}[\gamma^*] =$$

$$= \mathrm{Sup}_g\, A_{p'}\, \mathfrak{N}_p[c]\, \mathfrak{U}_{p'}[g^*] \leqslant \mathrm{Sup}_g\, A_{p'}\, \mathfrak{N}_p[c]\, \mathfrak{U}_{p'}[g] \leqslant A_{p'}\, \mathfrak{N}_p[c].$$

Since $\mathfrak{N}_p[c] < \infty$ involves $\mathfrak{N}_2[c] < \infty$, there is a sequence $\{f_{N_k}(x)\}$ which converges almost everywhere to $f(x)$, and so $f_{N_k}^*(x) \to f^*(x)$ for almost every x. Comparing the extreme terms of (5) and putting $N = N_k$, we obtain (2) by an application of Fatou's lemma.

The reader will easily convince himself that $A_q \leqslant M^{(q-2)/q} \alpha_q$ and $A'_p \leqslant M^{(2-p)/p} \alpha'_p$, where α_q depends only on q, and α'_p only on p.

9.5. Theorems of Hardy and Littlewood [1]). The theorems established in the previous paragraph are extensions to general orthogonal systems of results which had been obtained previously for the system $1, e^{ix}, e^{-ix}, e^{2ix}, \ldots$ by Hardy and Littlewood. This special case, however, is of independent interest, for the results may be stated in a different form and give the solution of an interesting problem. It will be convenient to change the notation of the previous paragraph slightly.

Given a sequence $c_0, c_1, c_{-1}, c_2, c_{-2}, c_3, \ldots$ let $c_0^* \geqslant c_1^* \geqslant c_{-1}^* \geqslant c_2^* \geqslant \ldots$ be the sequence $|c_0|, |c_1|, |c_{-1}|, \ldots$ arranged in the descending order of magnitude. Similarly, given a function $f(x)$, $-\pi \leqslant x \leqslant \pi$, we shall denote by $f^*(x)$, $-\pi \leqslant x \leqslant \pi$, the function which is equimeasurable with $|f(x)|$ and even; for $0 \leqslant x \leqslant \pi$, $f^*(x)$ may be defined as the function inverse to $\frac{1}{2}|\mathrm{E}(|f| \geqslant y)|$. We put

$$(1)\quad \mathfrak{B}_r[c] = \left\{\sum_{n=-\infty}^{+\infty} |c_n|^r (|n|+1)^{r-2}\right\}^{1/r}, \quad \mathfrak{U}_r[f] = \left\{\int_{-\pi}^{\pi} |f|^r |x|^{r-2}\, dx\right\}^{1/r}.$$

If, for a moment, we denote the sequence $c_0^*, c_1^*, c_{-1}^*, c_2^*, \ldots$ by $d_1^*, d_2^*, d_3^*, \ldots$, then the ratio $\sum_{-\infty}^{+\infty} c_n^{*r} (|n|+1)^{r-2} : \sum_{1}^{\infty} d_n^{*r} n^{r-2}$ is contained between two positive numbers depending exclusively on r. Thence we see that Theorems 9.4 remain true for the system $1, e^{ix}, e^{-ix}, \ldots$ if $\mathfrak{B}_r$ is given by the first formula (1). Similarly Theorems 9.43 are true for this system if the interval $(0, h)$ is replaced by $(-\pi, \pi)$ and $\mathfrak{U}_r$ is defined by the second formula (1).

We know that a necessary and sufficient condition that a sequence $c_0, c_1, c_{-1}, \ldots$ should be that of Fourier coefficients of an $f \in L^2$ is that $\Sigma |c_n|^2 < \infty$. This condition bears on the moduli of the c_n, so that a necessary and sufficient condition that the numbers $c_0, c_1, c_{-1}, \ldots$ should be, *for every variation* of their arguments, the Fourier coefficients of an $f \in L^2$, is again $\Sigma |c_n|^2 < \infty$. We ask whether anything similar is true for other classes L^r. The answer is negative: there can be no such condition for $r \neq 2$. For let us consider the series

---

[1]) Hardy and Littlewood [10], [15]; see also Gabriel [1], Mulholland [1].

(2) $$\sum_{n=1}^{\infty} n^{-\alpha} e^{inx}, \qquad \sum_{n=1}^{\infty} \pm n^{-\alpha} e^{inx} \qquad (0 < \alpha < 1).$$

If $\alpha = {}^3/_4$, the first series belongs to L^q if $q < 4$ only (§ 5.7.3), while the second belongs, for a special sequence of signs, to every L^q (§§ 5.6, 5.61) so that two functions, one of which belongs to L^q while the other does not, may have the same $|c_n|$. If $\alpha = {}^1/_4$, the first series in (2) belongs to L^p for $p < {}^4/_3$, while the second need not be a Fourier series.

These facts suggest a change in the problem. Now we shall vary not only the arguments of the c_n but also their order, and we ask when the new sequences will be those of Fourier coefficients, with respect to the system $1, e^{ix}, e^{-ix}, \ldots$, of functions belonging to L^r.

(i) *A necessary and sufficient condition that the c_n should be, for* e v e r y *variation of their arguments and arrangement, the Fourier coefficients of a function $f \in L^q$, is that $\mathfrak{B}_q[c^*] < \infty$; and then*

(3) $$\mathfrak{M}_q[f] \leqslant A_q \mathfrak{B}_q[c^*]$$

for every such f, where A_q depends on q only.

(ii) *A necessary and sufficient condition that the c_n should be, for* s o m e *variation of their arguments and arrangement, the Fourier coefficients of an $f \in L^p$, is that $\mathfrak{B}_p[c^*] < \infty$; and then*

(4) $$\mathfrak{B}_p[c^*] \leqslant A_p' \mathfrak{M}_p[f]$$

for every such f, where A_p' depends on p only.

For the proof we shall require the following lemmas.

9.501. (i) *If $a_1 \geqslant a_2 \geqslant \ldots \to 0$, a necessary and sufficient condition that the function $g(x) = \Sigma a_n \cos nx$ should belong to L^r, $r > 1$, is that the expression $S_r = \Sigma a_n^r n^{r-2}$ should be finite*

(ii) *The result remains true for sine series.*

Let $G(x)$ denote the integral of g, and $H(x)$ the integral of $|g|$, over $(0, x)$; let $A_n = a_1 + a_2 + \ldots + a_n$. By $B_1, B_2, \ldots$ we shall denote positive numbers which are either absolute constants or depend on r only. If $g \in L^r$, the series defining g is $\mathfrak{S}[g]$ (this is a corollary of the following proposition which will be established in Chapter XI: if a trigonometrical series converges, except at a finite number of points, to an integrable function f, the series is $\mathfrak{S}[f]$), and so

$$G(x)=\int_0^x g(t)\,dt=\sum_{n=1}^{\infty}\frac{a_n}{n}\sin nx,\quad G\left(\frac{\pi}{n}\right)=$$

$$=\sum_{m=1}^{n-1}\left(\frac{a_m}{m}-\frac{a_{m+n}}{m+n}+\frac{a_{m+2n}}{m+2n}-\dots\right)\sin\frac{m\pi}{n}\geqslant\sum_{m=1}^{n-1}\left(\frac{a_m}{m}-\frac{a_{m+n}}{m+n}\right)\sin\frac{m\pi}{n}\geqslant$$

$$\geqslant B_1\sum_{[n/3]+1}^{[2n/3]}\left(\frac{a_m}{m}-\frac{a_{m+n}}{m+n}\right)\geqslant B_2\sum_{[n/3]+1}^{[2n/3]}\frac{a_m}{m}\geqslant B_3\,a_n,$$

$$\sum_{n=2}^{\infty}a_n^r n^{r-2}\leqslant B_4\sum_{n=2}^{\infty}n^{r-2}\,G^r\left(\frac{\pi}{n}\right)\leqslant B_4\sum_{n=2}^{\infty}n^{r-2}H^r\left(\frac{\pi}{n}\right)\leqslant$$

$$\leqslant B_5\sum_{n=2}^{\infty}\int_{\pi/n}^{\pi/(n-1)}\left\{\frac{H(x)}{x}\right\}^r dx=B_5\int_0^{\pi}\left\{\frac{H(x)}{x}\right\}^r dx\leqslant B_6\int_0^{\pi}|f|^r\,dx$$

(§ 4.17, $s=0$) and the necessity of the condition in (i) is established. To show that the condition is sufficient we observe that

$$|g(x)|\leqslant\left|\sum_{\nu=1}^{n}a_\nu\right|+\left|\sum_{\nu=n+1}^{\infty}a_\nu\cos\nu x\right|\leqslant A_n+\frac{\pi a_n}{x}$$

(§ 1.22), and so $|g(x)|\leqslant B_7 A_n$ if $\pi/(n+1)\leqslant x\leqslant\pi/n$. Hence

(1) $$\int_0^{\pi}|g|^r\,dx=\sum_{n=1}^{\infty}\int_{\pi/(n+1)}^{\pi/n}|g|^r\,dx\leqslant B_8\sum_{n=1}^{\infty}A_n^r\,n^{-2},$$

and it remains to show that the last series converges whenever $S_r<\infty$. Let $f(x)$ denote the function which is equal to a_n for $n-1\leqslant x<n$, $n=1,2,\dots$, and let $F(x)$ be the integral of f over $(0,x)$. $S_r<\infty$ implies that $f^r(x)\,x^{r-2}\in L(0,\infty)$, and so, by Theorem 4.17 with $s=r-2$, $\{F(x)/x\}^r x^{r-2}=F^r(x)\,x^{-2}\in L(0,\infty)$. Since the last relation is equivalent to the convergence of the series $\Sigma A_n^r n^{-2}$, lemma (i) follows. Lemma (ii) may be obtained by a similar argument, or, still simpler, may be deduced from (i) using Theorem 7.21.

9.502. Now we are in a position to prove Theorems 9.5. That the condition of Theorem 9.5(i) is sufficient follows from Theorem 9.4(i), whence we also deduce the inequality 9.5(3). To prove that the condition is necessary, consider the series $\Sigma c_n^* e^{inx}$ and $\Sigma c_{-n}^* e^{inx}$. If both of them belong to L^q, so does their sum

$$\sum_{n=-\infty}^{+\infty}(c_n^*+c_{-n}^*)\,e^{inx}=2\left[c_0^*+\sum_{n=1}^{\infty}\tfrac{1}{2}\,(c_n^*+c_{-n}^*)\cos nx\right],$$

and from § 9.501(i) we obtain $\mathfrak{P}_q[c^*]<\infty$.

Theorem 9.4(ii) shows that the condition of Theorem 9.5(ii) is necessary. That it is also sufficient follows from the fact that the series $\sum_{n=-\infty}^{+\infty} c_n^* e^{inx}$ belongs to L^p if $\mathfrak{P}_p[c^*]<\infty$ (§ 9.501).

9.51 [1]**.** The following two theorems, in which we consider 'rearrangements' not of the Fourier coefficients but of the values the function, are, in a sense, reciprocals of Theorems 9.5.

(i) *A necessary and sufficient condition that* $\mathfrak{N}_q[c]$ *should be finite for* a l l $f(x)$ *having the same* $f^*(x)$, *is that* $\mathfrak{U}_q[f^*]$ *should be finite, and then*

(1) $$\mathfrak{N}_q[c]\leqslant A_q\,\mathfrak{U}_q[f^*].$$

(ii) *A necessary and sufficient condition that* $\mathfrak{N}_p[c]$ *should be finite for* s o m e $f(x)$ *with a given* $f^*(x)$, *is that* $\mathfrak{U}_p[f^*]$ *should be finite, and then*

(2) $$\mathfrak{U}_p[f^*]\leqslant A_p'\,\mathfrak{N}_p[c].$$

The proofs of (i) and (ii) are similar to those of Theorems 9.5 and are even a little easier since $f^*(x)$, unlike c_n^*, is a symmetrical function of its argument. The only thing we need is the following lemma: *if a function* $g(x)$, $|x|\leqslant\pi$, *is non-negative, even, and decreases in* $(0,\pi)$, *and if* a_n *are the cosine coefficients of* g, *then a necessary and sufficient condition that* $\mathfrak{N}_r[a]<\infty$, $r>1$, *is that the function* $g^r(x)\,x^{r-2}$ *should be integrable.* We shall only sketch the proof which runs on the same line as in § 9.501. Denoting by $G(x)$ the integral of g over $(0,x)$, we shall show that

(3) $$|a_n|\leqslant 2\,G\left(\frac{\pi}{n}\right),\quad A_n\geqslant B_{10}\,g\left(\frac{\pi}{n}\right),$$

where $A_n=|a_0|+|a_1|+\dots+|a_n|$. The first inequality follows from the formula

$$\frac{\pi}{2}\,a_n=\int_0^{\pi/n}g(x)\cos nx\,dx+\int_{\pi/n}^{\pi}g(x)\cos nx\,dx,$$

---

[1]) Hardy and Littlewood [10], [15].

where the last term on the right is, by the second mean-value theorem, less than $g(\pi/n)\cdot(2/n)\leqslant G(\pi/n)$ in absolute value. To prove the second inequality we notice that $\frac{1}{2}a_0+a_1+\ldots+a_{n-1}+\frac{1}{2}a_n$ is equal to

$$\frac{2}{\pi}\int_0^{\pi} g(t)\frac{\sin nt}{2\operatorname{tg}\frac{1}{2}t}dt \geqslant \frac{2}{\pi}\int_0^{\pi/n}\left[\frac{g(t)}{2\operatorname{tg}\frac{1}{2}t}-\frac{g(t+\pi/n)}{2\operatorname{tg}\frac{1}{2}(t+\pi/n)}\right]\sin nt\,dt \geqslant$$

$$\geqslant B_9\int_0^{\pi/2n}\frac{g(t)}{t}\sin nt\,dt \geqslant B_{10}\,g\left(\frac{\pi}{2n}\right)\geqslant B_{10}\,g\left(\frac{\pi}{n}\right).$$

Now it is sufficient to observe that, if $g^r x^{r-2}$ is integrable, so is $G^r(x)x^{-2}$, hence $\Sigma G^r(\pi/n)<\infty$, and, in view of the first inequality in (3), $\mathfrak{N}_r[a]<\infty$. Conversely, if $\mathfrak{N}_r[a]<\infty$, then $\Sigma\{A_n/n\}^r<\infty$, (this is an easy consequence of Theorem 4.17 with $s=0$) and the second inequality in (3) gives $\Sigma n^{-r}g^r(\pi/n)<\infty$. Since $g(x)$ is non-increasing we obtain that $g^r(x)x^{r-2}$ is integrable.

9.6. Banach's theorems on lacunary coefficients [1]. We know that a necessary condition for a sequence $\{a_n, b_n\}$ to be that of the Fourier coefficients of an integrable function f, is $|a_n|+|b_n|\to 0$. If a_n, b_n are to be the Fourier coefficients of a continuous f, the series $a_1^2+b_1^2+a_2^2+b_2^2+\ldots$ must converge. The converse propositions are obviously false, but we will prove that, at least for some values of n, the Fourier coefficients of integrable, or continuous, functions may be prescribed, roughly speaking, arbitrarily.

(i) *Let $\{n_i\}$ be any sequence of positive integers such that $n_{i+1}/n_i>\lambda>1$, $i=1,2,\ldots$, and let $\{x_i,y_i\}$ be an arbitrary sequence such that $(x_1^2+y_1^2)+(x_2^2+y_2^2)+\ldots<\infty$. Then there exists a continuous f with Fourier coefficients a_n, b_n satisfying the equations $a_{n_i}=x_i$, $b_{n_i}=y_i$, $i=1,2,\ldots$*

(ii) *If $\{n_i\}$ satisfies the same conditions as above and if $x_i\to 0$, $y_i\to 0$, there exists an integrable f such that $a_{n_i}=x_i$, $b_{n_i}=y_i$, $i=1,2,\ldots$*

We begin the proof of (i) by two remarks.

---

[1] Banach [1], Szidon [3], [4], Verblunsky [2].

(a) It is sufficient to prove the existence of a *bounded* f with the prescribed coefficients. For let $\{\varepsilon_k\}$ be a convex sequence tending to 0 and such that the series with terms $(x_i^2+y_i^2)/\varepsilon_{n_i}^2$ converges[1]). If we can find a bounded function $g \sim \frac{1}{2}a_0+(a_1\cos x+b_1\sin x)+\dots$, such that $a_{n_i}=x_i/\varepsilon_{n_i}$, $b_{n_i}=y_i/\varepsilon_{n_i}$, then $\frac{1}{2}a_0\,\varepsilon_0+(a_1\cos x+b_1\sin x)\,\varepsilon_1+\dots$ is the Fourier series of a continuous function (§ 4.65), and the terms with indices n_i in this series are $(x_i\cos n_i x+y_i\sin n_i x)$.

(b) It is sufficient to prove that, for every integer $k>0$, there exists a function $f_k(x)\sim\frac{1}{2}a_0^k+(a_1^k\cos x+b_1^k\sin x)+\dots$, such that $a_{n_i}^k=x_i$, $b_{n_i}^k=y_i$, $1\leqslant i\leqslant k$, and that $|f_k(x)|<C$, where C is a constant independent of k. In fact, let us assume, as we may, that $a_0^j=0$, $j=1,2,\dots$, and let $F_k(x)$ be the integral of f_k over $(0,x)$. Since the f_k are uniformly bounded, the functions F_k are uniformly continuous and we may find a subsequence $\{F_{m_k}\}$ converging uniformly to an $F(x)\in\mathrm{Lip}\,1$. The Fourier coefficients of F are limits of the corresponding Fourier coefficients of F_{m_k} as $k\to\infty$, and so the bounded function $f(x)=F'(x)$ has the prescribed coefficients for all the indices n_i.

Now we shall prove a number of lemmas.

9.601. *If $n_{k+1}/n_k>\lambda>1$, and if the series $\sum(a_k^2+b_k^2)$ converges, then*

$$(1)\qquad \sum_{k=1}^{\infty}(a_k\cos n_k x+b_k\sin n_k x)$$

is the Fourier series of a function $f(x)$ belonging to every class L^r, and

$$(2)\qquad \left\{\frac{1}{\pi}\int_0^{2\pi}|f(x)|^r\,dx\right\}^{1/r}\leqslant A_{r,\lambda}\left\{\sum_{k=1}^{\infty}(a_k^2+b_k^2)\right\}^{1/2},$$

where $A_{r,\lambda}$ depends only on r and λ.

This lemma will be required only in the case $r=4$, but the proof does not become simpler by considering any special value of r. Since the left-hand side of (2), multiplied by $2^{-1/r}$, is an increasing function of r (§ 4.15), it is sufficient to consider the

[1]) We may find first a sequence $\{\varepsilon'_k\}$, $\varepsilon'_k>0$, tending to 0 and then majorise it by a convex $\{\varepsilon_k\}$.

values $r = 2h$, $h = 1, 2, \dots$ Suppose first that the series (1) converges absolutely, and let $F(z) = \Sigma c_k z^{n_k}$ be the power series the real part of which, for $z = e^{ix}$, is (1). Then

$$F^h(z) = \sum_{\nu=0}^{\infty} d_\nu z^\nu,$$

where $d_\nu = 0$ if ν is not of the form

$$(3)\quad \alpha_1 n_{k_1} + \alpha_2 n_{k_2} + \dots, \quad \text{with } n_{k_1} > n_{k_2} > \dots, \quad \alpha_i > 0, \quad \alpha_1 + \alpha_2 + \dots = h.$$

Now we observe that, if λ is sufficiently large, $\lambda \geqslant \lambda_0$, then every positive integer can be represented at most once in the form (3). For otherwise we should have an equation $\beta_1 n_{k_1} + \beta_2 n_{k_2} + \dots = 0$, where $n_{k_1} > n_{k_2} > \dots$, $0 \leqslant |\beta_i| \leqslant h$, $\beta_1 \neq 0$, and so also $n_{k_1} \leqslant h(n_{k_2} + n_{k_3} + \dots)$, $1 < h(\lambda^{-1} + \lambda^{-2} + \dots)$, which is impossible if $\lambda \geqslant \lambda_0 = h + 1$.

By Parseval's theorem, $\frac{1}{2\pi}\int_0^{2\pi} |F^h(e^{ix})|^2 dx = \sum_{\nu=0}^{\infty} |d_\nu|^2$, where, if ν is of the form (3),

$$d_\nu = \frac{h!}{\alpha_1!\,\alpha_2!\,\dots} c_{k_1}^{\alpha_1} c_{k_2}^{\alpha_2} \dots, \qquad |d_\nu|^2 \leqslant h! \frac{h!}{\alpha_1!\,\alpha_2!\,\dots} |c_{k_1}|^{2\alpha_1} |c_{k_2}|^{2\alpha_2} \dots$$

Hence, if $\lambda \geqslant \lambda_0$, $\frac{1}{2\pi}\int_0^{2\pi} |F(e^{ix})|^{2h} \leqslant h! \left(\sum_{k=1}^{\infty} |c_k|^2\right)^h$, and since we have $|f(x)| \leqslant |F(e^{ix})|$, $c_k = a_k - ib_k$, the inequality (2) follows with $A_{2h,\lambda}^{2h} = 2h!$

To remove the condition concerning the absolute convergence of (1), we apply (2) to the function $f(r, x) = \Sigma (a_k \cos n_k x + b_k \sin n_k x) r^{n_k}$ and then make $r \to 1$.

To prove (2) for general $\lambda > 1$, we break up (1) into a finite number, say s, of series, for each of which the number λ is $\geqslant h + 1$. Correspondingly $f = f_1 + f_2 + \dots + f_s$. Since

$$\mathfrak{M}_{2h}[f_i] \leqslant (2h!)^{1/2h} \left\{\sum_{k=1}^{\infty} (a_k^2 + b_k^2)\right\}^{1/2}, \qquad \mathfrak{M}_{2h}[f] \leqslant \sum_{i=1}^{s} \mathfrak{M}_{2h}[f_i],$$

we obtain (2) with $A_{2h,\lambda} = s\,(2h!)^{1/2h}$.

9.602. *Under the conditions of the preceding lemma,*

$$(1)\qquad \frac{1}{\pi}\int_0^{2\pi}|f(x)|\,dx \geqslant B_\lambda \left\{\sum_{k=1}^{\infty}(a_k^2+b_k^2)\right\}^{1/2},$$

where B_λ depends only on λ.

If J_r denotes the left-hand side of 9.601(2) then, by Hölder's inequality, $J_2 \leqslant J_1^{1/3} J_4^{2/3}$ and so $J_1 \geqslant J_2^3/J_4^2$.

To prove (1) we apply the preceding lemma and observe that $A_{2,\lambda} = 1$.

9.603. Let $n_1, n_2, \ldots, n_k, \ldots$ be the sequence of Theorem 9.6(i). Let us fix an integer $k > 0$ and let B denote the set of all periodic functions f, $|f| \leqslant 1$. We put

$$x_i = \frac{1}{\pi}\int_0^{2\pi} f\cos n_i x\,dx, \qquad y_i = \frac{1}{\pi}\int_0^{2\pi} f\sin n_i x\,dx, \qquad 1 \leqslant i \leqslant k,$$

and denote by E the set, situated in the $2k$-dimensional space, of points $P(x_1, y_1, \ldots, x_k, y_k)$ obtained in this way. This set is convex, that is, if two points P_1, P_2 belong to it, so does every point $tP_1 + (1-t)P_2$, $0 \leqslant t \leqslant 1$, of the segment $P_1 P_2$. An argument similar to that used in § 9.6(b) shows that E is closed. We will now prove the following lemma.

E contains a whole 'sphere' $x_1^2 + y_1^2 + \ldots + x_k^2 + y_k^2 \leqslant R^2$, where $R = R_\lambda$ is a constant depending on λ but not on k.

Let $\alpha_1, \beta_1, \ldots, \alpha_k, \beta_k$ be an arbitrary set of numbers such that $\alpha_1^2 + \ldots + \beta_k^2 = 1$ and let

$$T(x) = (\alpha_1\cos n_1 x + \beta_1 \sin n_1 x) + \ldots + (\alpha_k \cos n_k x + \beta_k \sin n_k x).$$

If $P(x_1, \ldots, y_k)$ corresponds to an $f \in B$, we have the Parseval equation

$$(\alpha_1 x_1 + \beta_1 y_1) + \ldots + (\alpha_k x_k + \beta_k y_k) = \frac{1}{\pi}\int_0^{2\pi} f\,T\,dx.$$

For $f = \operatorname{sign} T \in B$ the last integral becomes $\pi^{-1}\mathfrak{M}[T] \geqslant R^2$, where $R = B_\lambda^{1/2}$ (§ 9.602). If we put $f = \theta \operatorname{sign} T$, where θ has a suitable value between 0 and 1, we obtain that $\alpha_1 x_1 + \ldots + \beta_k y_k = R^2$.

This fact may be interpreted geometrically[1]) as follows: on every 'plane' $\alpha_1 x_1 + \ldots + \beta_k y_k = R^2$, 'tangent' to the 'sphere' $(S_1) x_1^2 + \ldots + y_k^2 \leqslant R^2$, there exists a point $P \in E$.

Let us assume, contrary to what we intend to prove, that not all points of S_1 belong to E, and let P_0 be a point on the boundary of E nearest to the origin O. Let S_0 be the sphere with centre at the origin, having P_0 on its 'surface', P_1 the point where the radius OP_0 meets the surface of S_1, P a point belonging to E and situated on the plane Π_1 tangent to S_1 at P_1. It is obvious that $S_0 \subset E$, and that no point $Q \neq P_0$ on the segment $P_0 P_1$ belongs to E (for, otherwise, it would follow from the convexity of E that P_0 is a point interior to E[2])). The line PP_1 lies on Π_1, and so PP_0 cannot lie on the plane Π_0 tangent to S_0 at P_0 since Π_0 and Π_1 have no point in common. Thus the line PP_0 meets S_0 in more than one point. Thence we deduce, by continuity, that if $Q \neq P_0$ is a point on $P_0 P_1$ sufficiently near P_0, the line QP must have a point P_0' in common with S_0. It is easy to see that Q lies *between* P and P_0' (for P_0' and Q lie on different sides of Π_0), and since $P_0' \in E$, $P \in E$, so does Q. Here we have a contradiction since no point $Q \neq P_0$ on the segment $P_0 P_1$ belongs to E. This establishes the lemma.

9.604. Now we are in a position to prove Theorem 9.6(i). We put $(x_1^2 + y_1^2) + \ldots + (x_k^2 + y_k^2) = h_k^2$. From the last lemma follows the existence of a function $f_k(x)$, $|f_k(x)| \leqslant h_k/R$, such that the Fourier coefficients of f_k on the places n_i, $1 \leqslant i \leqslant k$, are equal to x_i, y_i. In virtue of remark (b) of § 9.6, this completes the proof of the theorem.

Corollary. Let $\varphi(u)$ be an arbitrary function tending to $+\infty$ with u. Then there exists a continuous function f having the Fourier coefficients a_n, b_n such that the series $\Sigma r_n^2 \varphi(1/r_n)$, where $r_n^2 = a_n^2 + b_n^2$, diverges[3]).

---

[1]) We use the geometrical language to make more intuitive the argument, which might be given a purely analytic form.

[2]) If P' is an arbitrary point situated sufficiently near to P_0, the line QP' meets S_0, and so $P' \in E$.

[3]) Gronwall [1], Szidon [4], Paley [3]. Putting $\varphi(u) = \log u$, we obtain an f such that $r_1^{2-\varepsilon} + r_2^{2-\varepsilon} + \ldots = \infty$ for every $\varepsilon > 0$ (§ 5.33).

For let $\{\alpha_k, \beta_k\}$ be an arbitrary sequence of numbers such that $\rho_1^2+\rho_2^2+\ldots<\infty$, $\rho_1^2\varphi(1/\rho_1)+\rho_2^2\varphi(1/\rho_2)+\ldots=\infty$, where $\rho_k^2=\alpha_k^2+\beta_k^2$. There exists a continuous f such that $a_{2^k}=\alpha_k$, $b_{2^k}=\beta_k$, say. Since $\rho_1^2\varphi(1/\rho_1)+\rho_2^2\varphi(1/\rho_2)+\ldots$ diverges, so does $r_1^2\varphi(1/r_1)+r_2^2\varphi(1/r_2)+\ldots$

9.61. The proof of Theorem 9.6(ii) is easier than that of Theorem 9.6(i) since we are able to give the required series explicitely [1]. First we prove the following lemma: *For any bounded sequence $\{x_i, y_i\}$ there exists a Fourier-Stieltjes series having x_i, y_i as the coefficients with the indices n_i.* It will be convenient to write x_{n_i}, y_{n_i} instead of x_i, y_i. We may suppose that $\rho_{n_i}^2=x_{n_i}^2+y_{n_i}^2\leqslant 1$. Let us assume first that $\lambda>3$. We put $x_{n_i}\cos n_i x+y_{n_i}\sin n_i x=$ $=\rho_{n_i}\cos(n_i x+\varphi_{n_i})$ and consider the partial products p_k of the product

$$(1)\qquad P=\prod_{i=1}^{\infty}\{1+\rho_{n_i}\cos(n_i x+\varphi_{n_i})\}.$$

Multiplying out these products, we see that no reduction of terms takes place (§ 6.4) and that the polynomial p_k is a partial sum of p_{k+1}. Making $k\to\infty$ we obtain, quite formally, a trigonometrical series. Since some partial sums, viz. p_k, are non-negative, this series is a Fourier-Stieltjes series (§ 4.39). Moreover the coefficients with suffixes n_i are x_{n_i}, y_{n_i}. It is important to observe that, if λ is large enough, $\lambda>\lambda_0(\varepsilon)$, the indices of terms different from 0 belong all to the intervals $(n_i(1-\varepsilon),\ n_i(1+\varepsilon))$, for every $0<\varepsilon<1$ (§ 6.4).

In the general case $\lambda>1$, we break up $\{n_i\}$ into r sequences $n_1^1, n_2^1, \ldots;\ n_1^2, n_2^2, \ldots;\ \ldots;\ n_1^r, n_2^r, \ldots$ in such a way that $n_{i+1}^s/n_i^s>\mu$, $i=1,2,\ldots$, $1\leqslant s\leqslant r$, $\mu>3$ being a large number which we shall define in a moment. Let P_s denote the product analogous to (1), consisting of factors $1+\rho_m\cos(mx+\varphi_m)$, where m runs through the sequence $n_1^s, n_2^s, \ldots$ We shall prove that $P_1+P_2+\ldots+P_r$ gives the required Fourier-Stieltjes series. In fact, if μ is large enough, the indices occuring in the series obtained from P_s all belong to the intervals $(n_i^s/\sqrt{\lambda},\ n_i^s\sqrt{\lambda})$, $i=1,2,\ldots$, so that the series $P_1, P_2, \ldots, P_r$ do not overlap. Since in the series P_s the terms with indices n_i^s have the coefficients $x_{n_i^s}$, $y_{n_i^s}$, the lemma follows.

---

[1]) Szidon [3].

To prove the theorem, let $\{\varepsilon_k\}$ be a convex sequence tending to 0 and such that $\{x_{n_i}/\varepsilon_{n_i}\}$ and $\{y_{n_i}/\varepsilon_{n_i}\}$ are bounded. If for a Fourier-Stieltjes series $\frac{1}{2}a_0 + (a_1 \cos x + b_1 \sin x) + \ldots$ we have $a_{n_i} = x_{n_i}/\varepsilon_{n_i}$, $b_{n_i} = y_{n_i}/\varepsilon_{n_i}$, the series $\frac{1}{2}a_0\varepsilon_0 + (a_1 \cos x + b_1 \sin x)\varepsilon_1 + \ldots$ is the required Fourier series (§§ 4.64, 5.12).

9.7. Wiener's theorem on functions of bounded variation. Let f be a function of bounded variation, a_n, b_n its Fourier coefficients, and $\rho_n^2 = a_n^2 + b_n^2$, $\rho_n \geqslant 0$. We know that, if f is discontinuous, then $n\rho_n \neq o(1)$ (§ 2.632), but since this inequality may occur also for f continuous (§ 5.7.14), it is not a necessary and sufficient condition for the discontinuity of f. It is interesting that such a condition may be obtained if the expressions $n\rho_n$ are replaced by their arithmetic means:

A necessary and sufficient condition that a function f of bounded variation be continuous is that $A_n = (\rho_1 + 2\rho_2 + \ldots + n\rho_n)/n \to 0$ [1].

We first prove the theorem in the following form: A necessary and sufficient condition for a function f of bounded variation to be continuous, is

(1) $$n\sum_{k=1}^{\infty} \rho_k^2 \sin^2 \frac{k\pi}{2n} \to 0 \quad \text{as } n \to \infty.$$

Let $\varphi_n(u) = [f(u+\pi/n) - f(u)]^2 + [f(u+2\pi/n) - f(u+\pi/n)]^2 + \ldots + [f(u+2\pi) - f(u+\pi(2n-1)/n)]^2$. Using Parseval's formula, we obtain

(2) $$\int_0^{2\pi} \varphi_n(u)\,du = 8\pi n \sum_{k=1}^{\infty} \rho_k^2 \sin^2 \frac{k\pi}{2n}$$

(§ 6.31). If f is continuous, $\omega(\delta)$ the modulus of continuity, and V the total variation of f, then, for every n, we have $\varphi_n(u) \leqslant \omega(\pi/n)\,V \to 0$ as $n \to \infty$, so that the right-hand side of (2) tends to 0, i. e. we have (1). Conversely, if f is discontinuous at a point ξ, $f(\xi+0) - f(\xi-0) = d \neq 0$, $2f(\xi) = f(\xi+0) + f(\xi-0)$, then, if n is large enough and (α, β) is any interval of length π/n containing ξ, we have $|f(\beta) - f(\alpha)| > d/3$. It follows that, if n is large, $\varphi_n(u) \geqslant d^2/9$ for every u and so the right-hand side of (2) does not tend to 0 as $n \to \infty$.

---

[1]) Wiener [2].

We shall now show that, if C_n is the left-hand side of (1), the relations $A_n \to 0$, $C_n \to 0$ are equivalent. Let B_n denote the ratio $(\rho_1^2 + 2^2\rho_2^2 + \dots n^2\rho_n^2)/n$. We shall show first that the relations $A_n \to 0$, $B_n \to 0$ are equivalent. Since the expressions $k\rho_k$ are bounded, the formula $A_n \to 0$ implies $B_n \to 0$. Applying Schwarz's inequality to the sum $1 \cdot \rho_1 + 1 \cdot 2\rho_2 + \dots + 1 \cdot n\rho_n$, we obtain that $A_n \leqslant B_n^{1/2}$, so that $B_n \to 0$ implies $A_n \to 0$.

It remains to prove that the relations $B_n \to 0$, $C_n \to 0$ are equivalent. Let us take only the first n terms in the series (1). Since $\sin u \geqslant 2u/\pi$ for $0 \leqslant u \leqslant \pi/2$, we see that $B_n \leqslant C_n$, and so, if $C_n \to 0$, then $B_n \to 0$. Observing that $\rho_k \leqslant \sqrt{2}\, V/k$ (§ 2.213) and breaking up the sum C_n into two, the first consisting of terms with indices $\leqslant nr$, where $r > 0$ is an integer, we see that

$$C_n \leqslant n \sum_{k=1}^{nr} \rho_k^2 \left(\frac{k\pi}{2n}\right)^2 + 2V^2 n \sum_{k=nr+1}^{\infty} \frac{1}{k^2}.$$

The first term on the right is equal to $B_{nr} \cdot \pi^2 r/4 \to 0$, if $B_n \to 0$. The second term is $< 2V^2/r$ and so is small for r large but fixed. This shows that $C_n \to 0$ if $B_n \to 0$, and the proof is complete.

9.8. Integrals of fractional order. Let $f(x)$ be integrable in an interval (a, b). Let $F_1(x)$ denote the integral of $f(t)$ over (a, x), $F_\alpha(x)$ the integral of $F_{\alpha-1}(t)$ over (a, x), $\alpha = 2, 3, \dots$ It can be verified by induction that

$$(1) \qquad F_\alpha(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt, \qquad a \leqslant x \leqslant b,$$

where $\Gamma(\alpha) = (\alpha-1)!$. If $\Gamma(\alpha)$ denotes the Euler Gamma function, the formula (1) may be taken as a definition of $F_\alpha(x)$ for every $\alpha > 0$. From the results of § 2.11 we deduce that $F_\alpha(x)$ exists for almost every x and is itself integrable [1]); for $\alpha \geqslant 1$ it is even continuous.

This definition of a fractional integral is due to Riemann and Liouville [2]). In the theory of periodic functions it is not entirely satisfatory since $F_\alpha(x)$ is not, in general, a periodic func-

---

[1]) For $\Gamma(\alpha)\, F_\alpha(x) = \int_a^b g(x-t) f(t)\, dt$, where $g(u) = u^{\alpha-1}$ for $u > 0$ and $g(u) = 0$ elsewhere.

[2]) Riemann [2], Liouville [1].

tion if f is one. Moreover it makes $F_\alpha(x)$ depend on a particular value of a. For this reason we shall consider another definition, propounded by Weyl, and more convenient in the theory of trigonometrical series [1]).

Let $f(x)$ be an integrable function having the period 1. (It simplifies the notation slightly if we consider functions of period 1 and not 2π, but this point is plainly without importance). *We assume that the mean value of f over* $(0, 1)$ *is equal to* 0, so that the constant term of $\mathfrak{S}[f]$ vanishes. It follows that the integral f_1 of f is also periodic, whatever the constant of integration. If we choose this constant of integration in such a way that the integral of f_1 over $(0, 1)$ vanishes, then the integral f_2 of f_1 will also be periodic, and so on. Generally, having defined the periodic functions $f_1, f_2, \dots f_{\alpha-1}$, we define $f_\alpha(x)$ as that of the primitives of $f_{\alpha-1}$, whose integral over $(0, 1)$ vanishes. Hence, the Fourier expansion of $f_\alpha(x)$ does not contain the constant term. In other words, if $f \sim \sum_{n=-\infty}^{+\infty} c_n e^{2\pi inx}$, $c_0 = 0$, then

$$(2) \qquad f_\alpha(x) = \sum_{n=-\infty}^{+\infty} c_n \frac{e^{2\pi inx}}{(2\pi in)^\alpha} = \int_0^1 f(t)\, \Psi_\alpha(x-t)\, dt,$$

$\Psi_\alpha(x)$ being the function which has the complex Fourier coefficients $\gamma_n^{(\alpha)} = (2\pi\, in)^{-\alpha}$, $\gamma_0 = 0$ (§ 2.15 [2]), where the actual function Ψ_α, corresponding to the interval $0 \leqslant x \leqslant 2\pi$, is denoted by f_k). The formula (2) may be considered as a definition of $f_\alpha(x)$ for every $\alpha > 0$, if we put $\gamma_n = (2\pi n)^{-\alpha} \exp(-\frac{1}{2}\alpha\pi i)$, $\gamma_{-n} = \overline{\gamma_n}$, $n > 0$, $\gamma_0 = 0$. From Theorem 5.12 we see that there really exists an integrable function $\Psi_\alpha(x)$ with Fourier coefficients γ_n. The integral in (2) exists for almost every x (§ 2.11), and the series converges almost everywhere. This last fact follows easily from the results of § 3.7, if we apply them not to the factors $1/\log n$ as in § 3.71 but to the factors $n^{-\alpha}$.

Let us denote $f_\alpha(x)$ by $I_\alpha[f]$. From (2) we see that $I_\beta[I_\alpha[f(x)]] = I_{\alpha+\beta}[f]$, $\alpha > 0$, $\beta > 0$. Since, for $\alpha = 1, 2, \dots$, $I_\alpha[f]$ coincides with the ordinary integral, the most interesting is the

---

1) Weyl [1].
2) See Errata.

case $0<\alpha<1$. To find the actual form of $\Psi_\alpha(x)$ we consider the formula

$$(3) \qquad \int_0^\infty t^{\alpha-1} e^{-it}\, dt = e^{-\frac{\pi i \alpha}{2}} \Gamma(\alpha), \qquad 0<\alpha<1,$$

which is easily obtained from the equation $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx$ by integrating round the contour

$$0<\varepsilon \leqslant z \leqslant R; \quad z = Re^{i\theta}, \quad 0 \leqslant \theta \leqslant \tfrac{1}{2}\pi; \quad z = ir, \quad R \geqslant r \geqslant \varepsilon;$$
$$z = \varepsilon e^{i\theta}, \quad \tfrac{1}{2}\pi \geqslant \theta \geqslant 0,$$

and then making ε and $1/R$ tend to 0. Making the substitution $t = 2\pi m u$ in (3), and taking into account the last remark of § 2.85, we see that, for $0<x<1$,

$$(4) \qquad \Gamma(\alpha)\, \Psi_\alpha(x) =$$
$$= \lim_{n\to\infty} \{x^{\alpha-1} + (x+1)^{\alpha-1} + \ldots + (x+n)^{\alpha-1} - n^\alpha/\alpha\}, \quad 0<\alpha<1.$$

It is easy to see that, if we omit the term $x^{\alpha-1}$ in the expression on the right, the limit, which we shall denote then by $\Gamma(\alpha)\, r_\alpha(x)$, exists uniformly in $0 \leqslant x \leqslant 1$. Taking this into account and observing that in the integral in (2) we may substitue $f(t)\, \Psi_\alpha(x-t)$ for $f(x-t)\, \Psi_\alpha(t)$, we obtain from (2) and (4) that

$$(5) \qquad f_\alpha(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty f(x-t)\, t^{\alpha-1}\, dt = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x f(t)\, (x-t)^{\alpha-1}\, dt.$$

It appears that the new definition differs from (1) in that the lower limit of integration is equal to $-\infty$. It must be remembered that the integrals (5) only converge owing to the fact that the mean value of f over (0, 1) vanishes.

Let $\Psi^*_\alpha(x)$, $-1<x<1$, be the function equal to 0 in $(-1, 0)$ and to $x^{\alpha-1}/\Gamma(\alpha)$ in (0, 1). Since $\Psi_\alpha(x+1) = \Psi_\alpha(x)$, considering the cases $-1<x\leqslant 0$ and $0<x<1$ separately, we see that $\Psi_\alpha(x) - \Psi^*_\alpha(x)$ is regular and equal to the function $r_\alpha(x)$ for $-1<x<1$. If we replace Ψ_α by Ψ^*_α in the integral (2), the function f_α is changed into F_α from (1) (with $a=0$). Thence we conclude that the function $f_\alpha(x) - F_\alpha(x)$ is regular for $0<x<1$, and so the two definitions of a fractional integral are, after all, not so essentially different.

It is easy to define *derivatives* $f^{\alpha}(x)$ of fractional order. For the sake of simplicity, we confine ourselves to the case $0<\alpha<1$, which is the most interesting in applications; and we put $f^{\alpha}(x)=\frac{d}{dx}f_{1-\alpha}(x)$. It is easy to prove that, if $f_{1-\alpha}(x)$ is absolutely continuous (in particular, if f^{α} is continuous), then $f(x)$ is the α-th integral of f^{α}. In fact, from the definition of f^{α} we see that $\mathfrak{S}[f^{\alpha}]$ is obtained by term-by-term differentation of $\mathfrak{S}[f_{1-\alpha}]$. In other words, $\mathfrak{S}[f^{\alpha}]$ may be obtained from $\mathfrak{S}[f]$ by introducing into the latter series the factors $\gamma_n^{(-\alpha)}=(2\pi in)^{\alpha}$, and this shows that f is the α-th integral of $f^{\alpha}(x)$.

9.81. Integration of functions satisfying Lipschitz conditions [1]. (i) *Let* $0\leqslant\alpha<1$, $\beta>0$, $\alpha+\beta\leqslant1$. *If* $f\in\text{Lip}\,\alpha$, *then* $f_{\beta}\in\text{Lip}\,(\alpha+\beta)$. (ii) *Let* $0<\gamma<\alpha\leqslant1$. *If* $f\in\text{Lip}\,\alpha$, *then* f^{γ} *exists and belongs to* $\text{Lip}\,(\alpha-\gamma)$.

Let $F(t)$ denote the integral of f over $(0,t)$, so that $F(x)-F(x-t)$ is a primitive function of $f(x-t)$ with respect to t. Integrating by parts the first integral in 9.8(5) and observing that $F(x)-F(x-t)$ vanishes for $t=0$, we obtain

$$(1)\qquad \Gamma(\beta)\,f_{\beta}(x)=(1-\beta)\int_0^{\infty}[F(x)-F(x-t)]\,t^{\beta-2}\,dt.$$

Let us write a similar equation for $f_{\beta}(x+h)$, $h>0$, and substract (1) from it. We have $\Gamma(\beta)\,[f_{\beta}(x+h)-f_{\beta}(x)]=A_h+B_h$, where A_h, B_h denote the integrals over $(0,h)$, (h,∞) respectively. The integrand of A_h may be represented in the form

$$(2)\qquad \begin{aligned}&(1-\beta)\,t^{\beta-2}\{[F(x+h)-F(x)]-[F(x+h-t)-F(x-t)]\}=\\ &=(1-\beta)\,t^{\beta-1}[f(x+h-\theta t)-f(x-\theta t)]\ ^{2)},\end{aligned}$$

where $\theta,\theta_1,\dots$ are numbers contained between 0 and 1. Since $|f(u_1)-f(u_2)|\leqslant M|u_1-u_2|^{\alpha}$, M denoting a constant, we find

---

[1] Hardy and Littlewood [6]. A special case of (ii) will be found in Weyl [1].

[2] Here we employ the mean-value theorem.

that (2) does not exceed $M(1-\beta)\,t^{\beta-1}h^{\alpha}$ in absolute value, and so $|A_h| \leqslant M(1-\beta)\,h^{\alpha+\beta}/\beta$.

The left-hand side of (2) may also be written in the form

$$(1-\beta)\,t^{\beta-2}\{[F(x+h)-F(x+h-t)]-[F(x)-F(x-t)]\} =$$
$$= (1-\beta)\,t^{\beta-2}h\,[f(x+\theta_1 h)-f(x+\theta_1 h-t)].$$

This expression does not exceed $M(1-\beta)\,t^{\alpha+\beta-2}h$ in absolute value and so $|B_h| \leqslant M h^{\alpha+\beta}(1-\beta)/(1-\alpha-\beta)$. Collecting the results we see that $|f_\beta(x+h)-f_\beta(x)| \leqslant M_1 h^{\alpha+\beta}$, where M_1 is independent of x and h. This completes the proof of (i).

If $\alpha+\beta=1$, it is not difficult to obtain that the modulus of continuity $\omega(\delta; f_\beta)$ of f_β is $O(\delta \log 1/\delta)$.

Passing to the proof of (ii), we observe that, since $f^{\gamma}(x)=\dfrac{d}{dx}f_{1-\gamma}$, we have to prove that $f_{1-\gamma}$ possesses a derivative belonging to Lip $(\alpha-\gamma)$. Let us put $\beta=1-\gamma$ in the formula (1); differentiating the integral on the right with respect to x, we obtain

$$\gamma\int_0^\infty [f(x)-f(x-t)]\,t^{-\gamma-1}\,dt. \tag{3}$$

Since $|f(x)-f(x-t)| \leqslant Mt^{\alpha}$ and f is bounded, the integral (3) converges uniformly in the neighbourhoods of $t=0$ and $t=\infty$, and so represents a continuous function $\varphi(x)$. It remains to show that $\varphi \in$ Lip $(\alpha-\gamma)$. Let us replace x by $x+h$, $h>0$, in (3) and substract (3) from the new integral. Breaking up the interval of integration $(0,\infty)$ into two, $(0,h)$ and (h,∞), we have, as in the proof of (i), $\varphi(x+h)-\varphi(x)=A_h+B_h$. The integrand in A_h does not exceed $\gamma t^{-\gamma-1}[|f(x+h)-f(x+h-t)|+|f(x)-f(x-t)|] \leqslant 2M\gamma\,t^{\alpha-\gamma-1}$ in absolute value, and, consequently, $|A_h| \leqslant 2Mh^{\alpha-\gamma}\gamma/(\alpha-\gamma)$. The integrand of B_h does not exceed $2M\gamma\,h^{\alpha}t^{-\gamma-1}$, and $|B_h| \leqslant 2Mh^{\alpha-\gamma}$. Hence $f^{\gamma} \in$ Lip $(\alpha-\gamma)$.

It has been proved by Hardy [4] that the Weierstrass series considered in § 2.9.3 is nowhere differentiable if $ab=1$. If $a=1/b$, that series may be considered as the $(1-\alpha)$-th integral of a trigonometrical series which is a linear combination of the series

$$\sum_{n=1}^{\infty} b^{-\alpha n}\cos b^n x, \qquad \sum_{n=1}^{\infty} b^{-\alpha n}\sin b^n x. \tag{4}$$

Each of the series (4) belongs to Lip α (for the first of them this was actually proved in § 2.9.3; the proof for the second remains essentially the same). This shows that the proposition (ii) is false for $\gamma = \alpha$: for a function $f \in \text{Lip}\,\alpha$, $0 \leqslant \alpha < 1$, there may be no point at which the derivative $f^{\alpha}(x)$ exists. The same example shows that proposition (i) fails for $\alpha + \beta = 1$.

9.82. Integration of functions belonging to a class L^p [1]. In the rest of this chapter we abandon our convention concerning the use of the letters p, q, which may now denote any numbers greater than 1.

(i) *If $f \in L^p$, $p > 1$, and $0 < \alpha < 1/p$, then $f_\alpha \in L^q$, where q is given by the formula $1/p - 1/q = \alpha$. Moreover $\mathfrak{M}_q[f_\alpha; 0, 1] \leqslant K \mathfrak{M}_p[f; 0, 1]$, where $K = K(p, q)$ depends only on p and q.*

(ii) *If $p > 1$, $1/p < \alpha < 1/p + 1$, then $f_\alpha \in \text{Lip}(\alpha - 1/p)$.*

We begin by proving (ii), which is comparatively easy. In virtue of Theorem 9.81(i), it is sufficient to consider the case $1/p < \alpha < 1$. Applying Hölder's inequality, we see that the left-hand side of the equation

$$f_\alpha(x + h) - f_\alpha(x) = \int_0^1 f(x - t)\,[\Psi_\alpha(t + h) - \Psi_\alpha(t)]\,dt$$

does not exceed $\mathfrak{M}_p[f]\,\mathfrak{M}_{p'}[\Psi_\alpha(t + h) - \Psi_\alpha(t)]$ in absolute value, and we have only to show that the second factor is $O(h^{\alpha - 1/p})$. Supposing that $0 < h < 1/2$, we may write

$$(1)\qquad \int_0^1 |\Psi_\alpha(t + h) - \Psi_\alpha(t)|^{p'}\,dt = \int_0^h + \int_h^{1-h} + \int_{1-h}^1 = P + Q + R.$$

Denoting by $C, C_1, \ldots$ constants which depend only on α and p, we may write the following inequalities, true for $0 < t \leqslant 1$ and $0 < \alpha < 1$:

$$(2)\qquad |\Psi_\alpha(t)| \leqslant C t^{\alpha - 1}, \qquad\qquad |\Psi'_\alpha(t)| \leqslant C_1 t^{\alpha - 2}.$$

The second of them is an immediate corollary of the formula $\Gamma(\alpha)\,\Psi'_\alpha(t) = \lim\,[t^{\alpha-2} + (t+1)^{\alpha-2} + \ldots + (t+n)^{\alpha-2}]$ $(n \to \infty)$ which, in turn, follows from 9.8(4). Returning to the equation (1) we see that, if $0 < t \leqslant h$, then $|\Psi_\alpha(t + h) - \Psi_\alpha(t)| \leqslant 2Ct^{\alpha-1}$, and so $P \leqslant C_2\,h^{(\alpha-1)p'+1}$

---

[1] Hardy and Littlewood [6]; see also Hardy, Littlewood, and Pólya, *Inequalities*, Chapter X.

(the reader will observe that $(\alpha-1)p' > -1$ since $\alpha > 1/p$). Similarly, since $R = \int_0^h |\Psi_\alpha(t) - \Psi_\alpha(t-h)|^{p'}\,dt$, and since for $0 < t \leqslant h$ we have $|\Psi_\alpha(t) - \Psi_\alpha(t-h)| = |\Psi_\alpha(t) - \Psi_\alpha(1+t-h)| \leqslant 2Ct^{\alpha-1}$, the expression R satisfies the same inequality as P. Finally, if $h \leqslant t \leqslant 1-h$, we obtain, by the mean-value theorem, that $|\Psi_\alpha(t+h) - \Psi_\alpha(t)| \leqslant C_1 h t^{\alpha-2}$ and so $Q \leqslant C_3 h^{p'} h^{(\alpha-2)p'+1}$. Collecting the results we see that $P+Q+R \leqslant C_4 h^{(\alpha-1)p'+1}$. Thus $\mathfrak{M}_p[\Psi_\alpha(t+h) - \Psi_\alpha(t)] = O(h^{\alpha-1/p})$. This completes the proof of the second part of the theorem.

Remarks. (a) Putting $f = f_1 + f_2$, where f_1 is a trigonometrical polynomial and $\mathfrak{M}_p[f_2]$ is very small, it is easy to see that $\omega(\delta; f_\alpha) = o(\delta^{\alpha-1/p})$.

(b) The theorem which we have proved holds also for $p=1$, $1 \leqslant \alpha < 2$. This follows from Theorem 9.81(i) and the fact that the integral of f is continuous

9.83. Theorem 9.82(i) is rather deep; its proof is long and will be based on a series on lemmas. Before we pass on to these lemmas we observe that a theorem less general than Theorem 9.82(i), viz. that $f_\alpha \in L^{q-\varepsilon}$ for every $\varepsilon > 0$, is trivially true. For $\Psi_\alpha(t) = O(t^{\alpha-1})$ in the neighbourhood of $t=0$, so that $\Psi_\alpha(t) \in L^{1/(1-\alpha)-\varepsilon}$, and we need only apply Theorem 4.16.

9.831. The first of the lemmas is as follows: *Let* $f(x) \geqslant 0$, $g(x) \geqslant 0$ *belong respectively to* $L^p(0,\infty)$, $L^q(0,\infty)$, *where* $p > 1$, $q > 1$. *If* $\lambda = 1/p + 1/q - 1 = 1 - 1/p' - 1/q' > 0$, $\mathfrak{M}_p[f; 0,\infty] = A$, $\mathfrak{M}_q[g; 0, \infty] = B$, *and if* $F(x)$ *denotes the integral of* f *over* $(0,x)$, *then*

$$(1) \qquad \int_0^\infty \frac{F(t)}{t}\, g(t)\, t^\lambda\, dt \leqslant K_1 AB, \qquad (K_1 = p'^{p/q'}).$$

Applying Hölder's inequality we see that the left-hand side does not exceed B multiplied by

$$(2) \qquad \left(\int_0^\infty F^{q'}\, t^{-\frac{q'}{p'}-1}\, dt\right)^{1/q'}.$$

From the inequality $1/p + 1/q > 1 = 1/q + 1/q'$ we see that $q' > p$. Hölder's inequality applied to the integral defining F gives $F(t) \leqslant At^{1/p'}$. Hence, writing $F^{q'} = F^{q'-p}F^p$, we see that (2) does not exceed

$$A^{\frac{q'-p}{q'}}\left(\int_0^\infty F^p\, t^{\frac{q'-p}{p'}}\, t^{-\frac{q'}{p'}-1}\, dt\right)^{1/q'} = A^{\frac{q'-p}{q'}}\left(\int_0^\infty \left(\frac{F}{t}\right)^p dt\right)^{1/q'},$$

and, by Theorem 4.17, the right-hand side does not exceed $Ap'^{p/q'}$.

9.832. The second lemma is: *Let f, g satisfy the conditions of the preceding lemma and be, in addition, non-increasing. Putting $\mu = 2 - 1/p - 1/q$, we have*

$$I = \int_0^\infty\int_0^\infty \frac{f(x)\, g(t)}{|x-t|^\mu}\, dx\, dt \leqslant K_2\, AB,$$

where K_2 depends only on p and q. Since

$$I = \int_0^\infty g(t)\, dt\left[\int_0^t f(x)(t-x)^{-\mu} dx\right] + \int_0^\infty f(x)\, dx\left[\int_0^x g(t)(x-t)^{-\mu} dt\right] = I_1 + I_2,$$

it suffices, in virtue of the symmetrical rôle of f and g, to consider e. g. I_1. Let $\lambda = 1 - \mu$; decomposing the inner integral in I_1 into two, taken over $(0, t/2)$ and $(t/2, t)$, and remembering that f is monotonic, we find that this integral does not exceed

$$(\tfrac{1}{2}t)^{-\mu} F(\tfrac{1}{2}t) + f(\tfrac{1}{2}t)\ (\tfrac{1}{2}t)^\lambda/\lambda \leqslant 2\lambda^{-1}(\tfrac{1}{2}t)^\lambda F(\tfrac{1}{2}t)/(\tfrac{1}{2}t) \leqslant 4\lambda^{-1} t^\lambda F(t)/t,$$

since $f(u) \leqslant F(u)/u$. It remains to apply the preceding lemma.

9.833. The third lemma, which is the most fundamental, may be enunciated as follows:

Let $f(x)$, $g(x)$, $h(x)$ be three non-negative functions defined in $(-\infty, +\infty)$. Let $f^*(x)$, $g^*(x)$, $h^*(x)$ denote three functions, even, non-increasing in $(0, \infty)$, and equimeasurable[1]) *with f, g, h respectively. If*

$$(1)\qquad I = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x)\, g(t)\, h(x+t)\, dx\, dt$$

and I^* is the corresponding integral formed with f^*, g^*, h^*, then $I \leqslant I^*$.

This lemma asserts that, among all functions equimeasurable with f, g, h, the maximum of I is attained when the functions are even and non-increasing in the interval $(0, \infty)$.

(i) We start with the case in which f, g, h are characteristic functions of sets F, G, H consisting of a finite number of inter-

---

[1]) § 9.42. Let $m(y) = |E(f > y)|$. We may define $f^*(x)$, $0 < x < \infty$, as the function inverse to $\frac{1}{2}m(y)$. We assume that $f^*(x) = f^*(x+0)$ for $x \geqslant 0$.

vals, each of the form $(n, n+1)$, $n = 0, \pm 1, \pm 2, \ldots$ We shall suppose that the numbers of these intervals in which f, g, h non vanish are $2\alpha, 2\beta, 2\gamma$, respectively, α, β, γ being even. Let

$$(2) \quad \varphi(x) = \int_{-\infty}^{+\infty} g(t)\, h(x+t)\, dt, \qquad \psi(x) = \int_{-\infty}^{+\infty} g^*(t)\, h^*(x+t)\, dt.$$

The continuous curves $y = \varphi(x)$, $y = \psi(x)$ are linear in the intervals $(n, n+1)$, and $y = 0$ for $|x|$ large. The function $\psi(x)$ is even, vanishes for $x \geqslant \gamma + \beta$, is equal to 2β for $0 \leqslant x \leqslant \gamma - \beta$ (assuming, as we may, that $\gamma \geqslant \beta$), and is linear in $(\gamma - \beta, \gamma + \beta)$; $\varphi(x)$ never exceeds 2β. Integrating (2) we find that the areas of the two curves are the same, viz. $4\beta\gamma$. Multiplying $\varphi(x)$ by $f(x)$, $\psi(x)$ by $f^*(x)$, and integrating over $(-\infty, +\infty)$, we deduce the lemma from geometrical considerations if $\alpha \leqslant \gamma - \beta$ or $\alpha \geqslant \gamma + \beta$.

Suppose then that $\gamma - \beta < \alpha < \gamma + \beta$. We can find two integers $\beta_0 < \beta$, $\gamma_0 < \gamma$ such that $\gamma_0 - \beta_0 = \gamma - \beta$, $\gamma_0 + \beta_0 = \alpha$. The lemma is true for $\alpha, \beta_0, \gamma_0$. Thence we will deduce it for $\alpha, \beta_0 + 1, \gamma_0 + 1$. For the values of $\psi(x)$ in the interval $(-\alpha, \alpha)$ will increase exactly by 2, and the result will be established when we have shown that the values of $\varphi(x)$ in $(-\infty, +\infty)$ will increase at most by 2. Since φ is linear in the intervals $(n, n+1)$, it suffices to consider integral values of x.

If H_x denotes the set H translated by x, then $\varphi(x) = |GH_x|$ represents the number of intervals of length 1 common to G and H_x. Now we may plainly suppose that one of the two intervals which we add to G (and similarly to H_x) is extreme on the left, and the other extreme on the right, with respect to G. Then the reader will easily convince himself that GH_x will increase by at most two intervals, each of length 1. For let J', J'' be the intervals which are added on the left to G and H_x respectively; then $(G + J')(H_x + J'') - GH_x = J'(H_x + J'') + GJ''$. If J' does not belong $H_x + J''$, then $|GH_x|$ remains unchanged when $|J''G| = 0$, and increases by 1 otherwise. If J' belongs to $H_x + J''$, then J'' lies to the left of G; hence $|J''G| = 0$ and $|GH_x|$ increases by 1.

The same argument gives the result for $\alpha, \beta_0 + 2, \gamma_0 + 2$, and so on, and finally for α, β, γ.

(ii) Changing variables we establish the truth of the lemma when the intervals have rational end-points. The restriction that the number of intervals in each set is divisible by 4 can now be removed, since, if this is not so, each of the intervals may be divided into four equal parts.

(iii) To prove the lemma in the case when F, G, H are arbitrary measurable sets, we observe that F (and similarly G, H) is a difference between an open set and a set of arbitrarily small measure; hence, for every $\varepsilon > 0$, we have $F = \mathcal{F} + F_1 - F_2$ where $\mathcal{F}$ consists of a finite number of intervals with, say, rational end-points, and $|F_1| < \varepsilon, |F_2| < \varepsilon$. The reader will have no difficulty in reducing the present case to the case (ii), observing that, roughly speaking, if one of the numbers $|F|, |G|, |H|$ is small, the integral I in (1) is small.

In the above argument we tacitly assumed that each of the numbers $|F|, |G|, |H|$ is finite. That the result holds without this assumption will follow from proposition (v) below.

(iv) If $f \geqslant 0$ is any function which only takes a finite number of values $\alpha_1, \alpha_2, \dots \alpha_m$, then $f = u_1 f_1 + u_2 f_2 + \dots + u_m f_m$, where $u_1, \dots, u_m$ are positive constants and $f_1, f_2, \dots, f_m$ are the characteristic functions of sets $F_1 \subset F_2 \subset \dots \subset F_m$. Then $f^* = u_1 f_1^* + u_2 f_2^* + \dots + u_m f_m^*$. If, in the same way, $g = v_1 g_1 + \dots + v_n g_n, h = w_1 h_1 + \dots + w_p h_p$, then

$$I = \Sigma\, u_i\, v_j\, w_k\, I_{ijk} \leqslant \Sigma\, u_i\, v_j\, w_k\, I^*_{ijk} = I^*,$$

where I_{ijk} are formed with f_i, g_j, h_k. This proves the lemma when f, g, h assume only a finite number of values.

(v) Let $\{f_n\}, \{g_n\}, \{h_n\}$ be three increasing sequences of non-negative functions and let $f_n \to f, g_n \to g, h_n \to h$. If the lemma is true for f_n, g_n, h_n, it is also true for f, g, h. In fact, $f_n(x)\, g_n(t)\, h_n(x+t)$ tends, increasing, to $f(x)\, g(t)\, h(x+t)$, and so, using an obvious notation, we have, by Lebesgue's theorem, $I_n \to I$. On the other hand, $f^* \geqslant f_n^*, g^* \geqslant g_n^*, h^* \geqslant h_n^*$; hence $I^* \geqslant I_n^* \geqslant I_n$ and, consequently, $I^* \geqslant I$.

(vi) Every non-negative function f is the limit of an increasing sequence of functions assuming only a finite number of values; e. g. we may put $f_n(x) = 2^{-n} k$, $0 \leqslant k < n 2^n$, where $k 2^{-n} \leqslant f_n(x) < (k+1) 2^{-n}$, and $f_n(x) = n 2^n$ elsewhere. From this and (iv), (v), we conclude the truth of the lemma in the general case.

Changing t into $-t$ in (1) we obtain a similar result for integrals (1) with $h(x-t)$ instead of $h(x+t)$.

9.84. Completion of the proof of Theorem 9.82(i). Let us replace, as we may, the interval of integration $(0,1)$ in the formula 9.8(2) by $(-\frac{1}{2},\frac{1}{2})$, and let $g(x)\in L^{q'}$ be an arbitrary periodic function such that $\mathfrak{M}_{q'}[g]=1$. Then (§ 4.7.2)

$$\mathfrak{M}_q[f_\alpha] = \operatorname*{Max}_g \int_{-1/2}^{1/2} f_\alpha(x)\,g(x)\,dx \leqslant \operatorname*{Max}_g \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} |f(t)\,g(x)\,\Psi_\alpha(x-t)|\,dx\,dt.$$

Let $f^*(x)$ be even, non-increasing in the interval $0<x<\infty$, and equimeasurable with the function equal to $|f(x)|$ for $|x|<\frac{1}{2}$ and to 0 elsewhere; similarly $g^*(x)$. Since $|\Psi_\alpha(u)|\leqslant C|u|^{\alpha-1}$ for $|u|<1$, where C depends only on α, we deduce from Lemma 9.833 (with $h(u)=|u|^{\alpha-1}$ and $h(x+t)$ replaced by $h(x-t)$) and Lemma 9.832 that $\mathfrak{M}_q[f_\alpha]$ does not exceed

$$\operatorname*{Max}_g 4C\int_0^\infty\int_0^\infty \frac{f^*(t)\,g^*(x)}{|x-t|^{1-\alpha}}\,dx\,dt \leqslant 4CK_2\,\mathfrak{M}_p[f^*;0,\infty]\,\mathfrak{M}_{q'}[g^*;0,\infty],$$

if $1-\alpha=2-1/p-1/q'$, i. e. if $\alpha=1/p-1/q$. Putting $4CK_2=K$ we obtain that $\mathfrak{M}_q[f_\alpha]\leqslant K\mathfrak{M}_p[f]$.

9.85. Theorem 9.82(i) is false for $p=1$, that is if $f\in L$, $q=1/(1-\alpha)$, then f_α need not necessarily belong to L^q. In fact, if $f(t)=-C+t^{-1}(\log 1/t)^{-1-1/q}$ for $0<t<1/2$, $f(t)=0$ for $1/2<t<1$, where C is a constant such that the mean value of f over $(0,1)$ vanishes, we have

$$f_\alpha(x)=\int_0^{1/2} f(t)\,\Psi_\alpha(x-t)\,dt=\int_0^{1/2} f(t)\,\Psi_\alpha^*(x-t)\,dt+R(x),$$

where R is a function regular in a neighbourhood of $x=0$, $\Psi_\alpha^*(u)=u^{\alpha-1}/\Gamma(\alpha)$ for $u>0$, $\Psi_\alpha^*(u)=0$ otherwise (§ 9.8). If $0<x<1/2$, the last integral exceeds

$$-\frac{Cx^\alpha}{\Gamma(\alpha+1)}+\frac{x^{\alpha-1}}{\Gamma(\alpha)}\int_0^x t^{-1}(\log 1/t)^{-1-1/q}\,dt.$$

Hence, for x small, $f_\alpha(x)\geqslant C_1x^{\alpha-1}(\log 1/x)^{-1/q}$, and so $f_\alpha \bar{\in} L^q$. To show that Theorem 9.81(i) is false for $\alpha=1/p$, i. e. that if $f\in L^p$, then $f_{1/p}$ need not be bounded, we may argue as follows. Multiplying the integral in 9.8(2) by $g(x)\in L$, integrating over $(0,1)$, and inverting the order of integration, we see that if, for every $f\in L^p$, $f_{1/p}$ were bounded, then, for every $g(x)\in L$, we should have $g_{1/p}\in L^{p'}$, which we know to be false.

9.86. It is of some interest to investigate whether Theorem 9.82(i) is a corollary of the theorems on Fourier coefficients established in the first part of this chapter. We shall show that this is really the case when $p\leqslant 2\leqslant q$,

p, q having the meaning of Theorem 9.82(i), and only then. Assuming f real, consider the inequalities

$$(1) \quad \sum_{n=1}^{\infty} c_n^{*p}\, n^{p-2} < \infty, \qquad (2) \quad \sum_{n=1}^{\infty} \frac{|c_n|^{q'}}{n^{\alpha q'}} < \infty,$$

where $c_1^*, c_2^*, \dots$ is the sequence $|c_1|, |c_2|, \dots$ rearranged in descending order of magnitude. The inequality (1) is implied by the relation $f \in L^p$, and (2) implies that $f_\alpha \in L^q$. Now (2) is certainly true if the series with terms $c_n^{*q'} n^{-\alpha q'}$ converges. We have $c_n^{*q'} n^{-\alpha q'} = c_n^{*p} n^{p-2} c_n^{*q'-p} n^{-\alpha q'+2-p}$ and, since

$$-\alpha q' + 2 - p = -(1/q' - 1/p')\, q' + 2 - p = q'/p' - p/p',$$

we obtain that $c_n^{*q'} n^{-\alpha q'} = c_n^{*p} n^{p-2} (c_n^* n^{1/p'})^{q'-p}$. Since the terms on the left in (1) decrease monotonically, the expression $c_n^{*p} n^{p-2} \cdot n$ is bounded, i. e. $c_n^* n^{1/p'} = O(1)$, and this, together with the last formula and the inequality (1), ensures the inequality (2), provided that $p \leqslant 2 \leqslant q$, $p \leqslant q'$. To get rid of the last condition assume that $p \leqslant 2 \leqslant q$ and $q' < p$. We have then $q' \leqslant 2 \leqslant p', q' < p$. Since $\alpha = 1/p - 1/q = 1/q' - 1/p'$, we see, by the result already obtained, that integration of order α transforms $L^{q'}$ into $L^{p'}$, and this is equivalent to the fact that the said integration transforms L^p into L^q (§ 4.63(ii))[1].

We have only proved that $\mathfrak{M}_q[f_\alpha] < \infty$, but in the same way we can obtain the complete result $\mathfrak{M}_q[f_\alpha] \leqslant K \mathfrak{M}_p[f]$.

It is easy to see why the above argument fails in the cases $p < q < 2$ or $2 < p < q$ (which are equivalent) e. g. in the latter. Integration of order α consists in introducing the factors $\gamma_n = |n|^{-\alpha} \varepsilon_n$ into $\mathfrak{S}[f]$, where $\{\varepsilon_n\}$ is a *special* sequence of unit numbers. The proof given above shows that, if $p \leqslant 2 \leqslant q$, the theorem holds when ε_n is an *arbitrary* bounded sequence. To show that such an extension is impossible for $2 < p < q$, let us suppose that the Fourier expansion of f is the cosine series with coefficients $\varepsilon_n / \sqrt{n} \log n$, $n = 2, 3, \dots$, where $\varepsilon_n = \pm 1$. Choosing for $\{\varepsilon_n\}$ a special sequence, we may have $f \in L^p$, $p > 2$ (§ 5.6). Introducing into $\mathfrak{S}[f]$ the factors ε_n / n^α, $0 < \alpha < \frac{1}{2}$, we obtain the series $\Sigma (\cos nx)/n^{\frac{1}{2}+\alpha} \log n$. In the neighbourhood of $x = 0$ the sum of this series behaves like $x^{-\frac{1}{2}+\alpha}/\log x$ and so it does not belong to L^γ if $\frac{1}{2} - 1/\gamma > \alpha$. If $\alpha = 1/p - 1/q < \frac{1}{2} - 1/q$, the series does not belong to L^q.

9.9. Miscellaneous theorems and examples.

1. Let $\omega_1(t), \omega_2(t), \dots, \omega_n(t)$ be a system of functions measurable and bounded in a finite interval $a \leqslant t \leqslant b$, and let

$$M_{\alpha\beta} = \sup_{x_1, \dots, x_n} \left\{ \int_a^b \left| \sum_{i=1}^n x_i\, \omega_i(t) \right|^{1/\beta} dt \right\}^\beta \Bigg/ \left\{ \sum_{i=1}^n |x_i|^{1/\alpha} \right\}^\alpha .$$

[1]) Theorem 4.63(ii) holds in the case of complex factors.

Show that (i) $M_{\alpha\beta}$ is a multiplicatively convex function in the triangle (Δ) $0 \leqslant \alpha \leqslant 1$, $0 \leqslant \beta \leqslant \alpha$, (ii) Theorem 9.11(b) is a consequence of (i). M. Riesz [3].

[Once the continuity of $M_{\alpha\beta}$ in the triangle Δ has been established, (i) may be proved by an argument similar to that of § 9.2, independently of the more difficult Theorem 9.23. To prove Theorem 9.11(b), we put $\omega_i(t) = \varphi_i(t)$, compute $M_{1,0}$ and $M_{1/2, 1/2}$, and obtain

(1) $$\mathfrak{M}_{p'}[s_n; a, b] \leqslant M^{(2-p)/p} \left\{ \sum_{i=1}^{n} |c_i|^p \right\}^{1/p},$$

where s_n is the n-th partial sum of the series (S) $c_1\varphi_1 + c_2\varphi_2 + \dots$ Since $\mathfrak{N}_2[c] < \infty$, S is the Fourier series of a function $f(t)$ and a subsequence $\{s_{n_k}\}$ of $\{s_n\}$ tends almost everywhere to $f(t)$. An application of Fatou's lemma to (1) completes the proof. If the interval where the functions φ_i are orthogonal is infinite, observe that the inequality (1) is true for any interval (a_1, b_1) completely interior to (a, b), and so it holds for (a, b) also].

2. Let $f(x)$ be a real function belonging to L^p, $1 < p \leqslant 2$, with Fourier coefficients a_n, b_n; the inequality of Theorem 9.1(a) then gives

$$\left\{ \left| \frac{a_0}{\sqrt{2}} \right|^{p'} + \sum_{n=1}^{\infty} (|a_n|^{p'} + |b_n|^{p'}) \right\}^{1/p'} \leqslant \left\{ \frac{1}{\pi} \int_0^{2\pi} |f(t)|^p \, dt \right\}^{1/p}$$

Inverting this inequality and interchanging the numbers p and p', we obtain the inequality corresponding to Theorem 9.1(b).

3. Let $1 < p \leqslant 2 \leqslant q$, $p \leqslant r \leqslant p'$, $q' \leqslant s \leqslant q$, $\lambda = 1/p + 1/r - 1$, $\mu = 1/q + 1/s - 1$. Then (i) Under the hypothesis of Theorem 9.4(ii),

(1) $$\left\{ \sum_{n=1}^{\infty} (c_n^* n^{-\lambda})^r \right\}^{1/r} \leqslant A'_p \, \mathfrak{M}_p[f],$$

where A'_p depends on p and M only.

(ii) If $\Sigma (c_n^* n^{-\mu})^s < \infty$, the series $\Sigma c_n \varphi_n$ is the Fourier series of a function $f \in L^q$, and

(2) $$\mathfrak{M}_q[f] \leqslant A_q \left\{ \sum_{n=1}^{\infty} (c_n^* n^{-\mu})^s \right\}^{1/s},$$

where A_q depends on q and M only.

The results are due, in substance, to Hardy and Littlewood [10], who considered the case of trigonometrical series.

[Proposition (i) is, so to speak, an intermediate result between Theorem 9.11(a) and Theorem 9.4(ii), and is a consequence of those theorems. To prove it, we observe that $r = t_1 p + t_2 p'$, $t_i \geqslant 0$, $t_1 + t_2 = 1$, apply Hölder's inequality to the left-hand side of (1) and use the theorems just quoted. To prove (2), we show that $(\Sigma c_n^{*q} n^{q-2})^{1/q}$ does not exceed $\{\Sigma (c_n^* n^{-\mu})^s\}^{1/s}$, and apply Theorem 9.4(i)].

4. Let $\{\varphi_n\}$ be a set of functions orthogonal, normal, and uniformly bounded ($|\varphi_n| \leqslant M$) in an interval (a, b). If $|c_n| \leqslant 1/n$, $n = 1, 2, \dots$, the c's are

the Fourier coefficients, with respect to $\{\varphi_n\}$, of a function f such that $\exp \lambda f$ is integrable for every $\lambda < 1/eM$.

[Assuming for simplicity that $c_1 = 0$, observe that

$$(*) \qquad \frac{\lambda^k}{k!}\int_a^b |f|^k\,dx \leqslant \frac{\lambda^k}{k!} M^{k-2}\left(\sum_{n=2}^{\infty} n^{-k/(k-1)}\right)^{k-1} < \frac{\lambda^k}{k!} M^{k-2}\frac{(k-1)^{k-1}}{k!}$$

for $k \geqslant 2$, and that $\exp \lambda u = 1 + \lambda u + \frac{1}{2}\lambda^2 u^2 + \dots$].

5. If the functions φ_n satisfy the conditions of the previous theorem, the interval (a, b) is finite, $|f|\log^+|f|$ is integrable over (a, b), and γ_n are the Fourier coefficients of f, then the series $\Sigma|\gamma_n|/n$ converges.

[This follows from the previous theorem by an application of Young's inequality. Observe that the inequality (*) holds if we replace f by any partial sum of the series $c_1\varphi_1 + c_2\varphi_2 + \dots$].

6. Under the conditions of the previous theorem we have

$$\sum_{n=1}^{\infty}\frac{\gamma_n^*}{n} < \infty, \qquad \sum_{n=1}^{\infty} e^{-k/|\gamma_n|} < \infty,$$

where $\{\gamma_n^*\}$ is the sequence $\{|\gamma_n|\}$ arranged in descending order of magnitude, and k is any positive number. For a similar result see Hardy and Littlewood [15].

[The second inequality follows from the first].

7. When $1 < p < 2$, equality in Theorem 9.1(a) occurs if and only if f is a trigonometrical monomial, i. e. if $f(x) = Ce^{inx}$, where C is a constant and $n = 0, \pm 1, \dots$ Similarly equality in Theorem 9.1(b) can occur only if all the c_n, except perhaps one, are equal to 0.

[For the proof (which is not quite simple) see Hardy and Littlewood [10]. The special case $p = 2k/(2k-1)$ is comparatively easy and may be proved by the argument of § 9.12, investigating cases of equality in Young's inequality 4.16(2). See also Hardy, Littlewood, and Pólya, *Inequalities*, Chapter VIII].

8. Let $P_1 = (\alpha_1, \beta_1)$ and $P_2 = (\alpha_2, \beta_2)$ be two points in the triangle (Δ) $0 \leqslant \alpha \leqslant 1$, $0 \leqslant \beta \leqslant \alpha$. If a sequence $\{\lambda_n\}$ belongs to $(L^{1/\alpha_1}, L^{1/\beta_1})$ and to $(L^{1/\alpha_2}, L^{1/\beta_2})$ (§ 4.6), then it belongs also to $(L^{1/\alpha}, L^{1/\beta})$ for every point (α, β) on the segment P_1P_2. M. Riesz [3].

[The proof follows the same line as in § 9.25].

9. (i) Let a_n, b_n be the Fourier coefficients of a function $f(x) \in L^p$, $p > 1$; then, if $n_{i+1}/n_i > \lambda > 1$, the series $\Sigma(a_{n_i}^2 + b_{n_i}^2)$ converges. More generally

(ii) If the power series $\Sigma\alpha_n z^n$ belongs to H (§ 7.51), the series $\Sigma|\alpha_{n_i}|$ converges. Paley [5]; Zygmund [5].

Proposition (i) shows that, if $\Sigma(x_i^2 + y_i^2) = \infty$, the function $f(x)$ of Theorem 9.6(ii) does not belong to any class L^p, $p > 1$.

[By Theorem 7.53(vi), $F(z) = F_1(z)\,F_2(z)$, where $F_1(z) = \Sigma\beta_n z^n$ and $F_2(z) = \Sigma\gamma_n z^n$ belong to H^2. Let $\Sigma|\beta_n|^2 = B^2$, $\Sigma|\gamma_n|^2 = C^2$. Then

$$|\alpha_{n_i}| \leqslant \left(\sum_{k=0}^{n_{i-1}} + \sum_{k=n_{i-1}+1}^{n_i}\right) |\beta_k \gamma_{n_i-k}| \leqslant B\left(\sum_{n_i-n_{i-1}}^{n_i} |\gamma_k|^2\right)^{1/2} + C\left(\sum_{n_{i-1}+1}^{n_i} |\beta_k|^2\right)^{1/2},$$

whence we easily deduce (ii)].

10. (i) If $0<\alpha<1$, $0<\beta<1$, $\alpha+\beta>1$, and if $f \in \text{Lip}\,\alpha$, then $f_\beta(x)$ has a derivative $f'(x) \in \text{Lip}\,(\alpha+\beta-1)$. (ii) If $0<\alpha<\gamma<1$, and if $f(x)$ has a derivative $f'(x) \in \text{Lip}\,\alpha$, then $f^\gamma(x) \in \text{Lip}\,(1+\alpha-\gamma)$.

[Corollaries of Theorems 9.81].

11. Theorem 9.82(i) holds for $p=1$ provided that $\tilde{\mathfrak{S}}[f]$ is a Fourier series.

For the proof, which is rather difficult, see Hardy and Littlewood [6_2].

12. Let $r>1$, $r'=r/(r-1)$. If $f \in L^r$, $g \in L^{r'}$, we have the formula 7.3(3), the series on the right being convergent. If $r=2$ the series converges absolutely. Show that this last result is false for any other value of r. M. Riesz [4].

[Suppose that $1<r<2$, and let $r<1/(1-\alpha)$, $0<\alpha<1/2$. There is a function $h(x) \in \text{Lip}\,\alpha$ such that $\mathfrak{S}[h]$ does not converge absolutely. We may assume that $\mathfrak{S}[h]=\Sigma a_n \cos nx$ is a purely cosine series, for otherwise, if x_0 is a point where $\mathfrak{S}[h]$ does not converge absolutely, we may consider $\frac{1}{2}[h(x_0+x)+h(x_0-x)]$ instead of $h(x)$. Let

$$f(x)=\sum_{n=1}^{\infty} n^{-\alpha}\cos nx, \qquad g(x)=h^\alpha(x)=\sum_{n=1}^{\infty} n^\alpha a_n \cos(nx+\tfrac{1}{2}\alpha\pi).$$

Since $\Sigma|a_n|=\infty$, the Parseval series for f and g does not converge absolutely, although $f \in L^r$ (§§ 5.221, 5.7.2), and g is continuous and so belongs to $L^{r'}$].

13. Let $f \sim \Sigma c_n e^{inx}$, $g \sim \Sigma d_n e^{inx}$. If $f \in L^p$, $g \in L^r$, where $1<p\leqslant 2$, $p \leqslant r \leqslant p'$, the series

$$\text{(1)} \qquad \sum_{n=-\infty}^{\infty} |n|^{-\lambda} e^{-\frac{1}{2}\lambda\pi i\,(\text{sign}\,n)} c_n d_n, \qquad \lambda = 1/p+1/r-1,$$

converges. If in addition $r \leqslant 2$, the series (1) converges absolutely. Hardy and Littlewood [14].

[If $r=p'$, the theorem follows from M. Riesz's equation 7.3(3). Applying this special case to the functions $f_\lambda(x)$ and $g(x)$, and taking account of Theorem 9.82(i), we obtain the convergence of (1). To obtain the second part of the theorem, apply Theorems 9.9.3(i) and 9.1(a)].

CHAPTER X.

Further theorems on the summability and convergence of Fourier series.

10.1. An extension of Fejér's theorem. Let $f(x)$ be an integrable and periodic function, and let $s_n(x)$ be the n-th partial sum of $\mathfrak{S}[f]$. Fejér's theorem asserts that, if f is continuous at the point x, then

$$\frac{1}{n+1}\sum_{\nu=0}^{n}\{s_\nu(x)-f(x)\}\to 0 \tag{1}$$

as $n\to\infty$. We shall prove a result from which it will follow in particular that, at every point of continuity of f,

$$\frac{1}{n+1}\sum_{\nu=0}^{n}|s_\nu(x)-f(x)|\to 0. \tag{2}$$

The relation (2) tells us that the mean value of $s_\nu(x)-f(x)$ tends to 0 not because of the interference of positive and negative terms, but because the indices ν for which $|s_\nu(x)-f(x)|$ is not small are comparatively sparse.

We shall require the following lemma.

If $f\in L^r$, $r\geqslant 1$, then, for almost every x, and h tending to 0,

$$\int_0^h |f(x\pm t)-f(x)|^r\,dx=o(h).$$

The case $r=1$ was considered in § 2.703, and the proof of the general result is not essentially different. For let α be any rational number, and let E_α be the set of x such that $\frac{1}{h}\int_0^h |f(x\pm t)-\alpha|^r\,dt$ does not tend to $|f(x)-\alpha|^r$ as $h\to 0$. Every set E_α, and so their sum E, is of measure 0. If $x \in E$ and if β is a rational number such that $|f(x)-\beta| < \frac{1}{2}\varepsilon$, then, by Minkowski's inequality,

$$\left\{\frac{1}{h}\int_0^h |f(x\pm t)-f(x)|^r\,dt\right\}^{1/r} \leqslant \left\{\frac{1}{h}\int_0^h |f(x\pm t)-\beta|^r\,dt\right\}^{1/r} + \\ + \left\{\frac{1}{h}\int_0^h |\beta - f(x)|^r\,dt\right\}^{1/r}.$$

Since the first term on the right tends to $|f(x)-\beta|$ as $h\to 0$, and the following term is equal to $|f(x)-\beta|$, the left-hand side of this inequality is less than ε for h sufficiently small. Since $\varepsilon > 0$ is arbitrary and $|E| = 0$, the lemma follows.

Let $\varphi_x(t) = f(x+t)+f(x-t)-2f(x)$; in view of the relation $|\varphi_x(t)| \leqslant |f(x+t)-f(x)| + |f(x-t)-f(x)|$, and applying Minkowski's inequality, we obtain that $\Phi_{x,r}(h) = \int_0^h |\varphi_x(t)|^r\,dt$ is $o(h)$ for almost every x. The chief object of this paragraph is the following theorem[1].

(i) *If $f \in L^r$, $r > 1$, and if k is any positive number, then, at every point x where $\Phi_{x,r}(h) = o(h)$, we have*

$$\text{(3)} \qquad \frac{1}{n+1}\sum_{\nu=0}^{n} |s_\nu(x) - f(x)|^k \to 0 \qquad \text{as } n\to\infty.$$

(ii) *If $f \in L$, and if f is continuous at every point of an interval $a \leqslant x \leqslant b$, the relation* (3) *holds uniformly in the interval* (a, b).

In the first place we observe that, if (3) is established for a certain value of k, it holds à fortiori for any smaller k; this follows from the fact that, if $c_1, c_2, \ldots c_m$ are arbitrary numbers, the expression $\{(|c_1|^k + |c_2|^k + \ldots + |c_m|^k)/m\}^{1/k}$ is a non-decreasing function of k (this expression is equal to $\mathfrak{A}_k[g; 0, m]$, where $g(x) = c_j$ for $j-1 < x \leqslant j$, $j = 1, 2, \ldots, m$; § 4.15). Secondly, it is sufficient to prove (3) for $k = r' = r/(r-1)$; for $\{\Phi_{x,r}(h)/h\}^{1/r}$ is a non-decreasing function of r and so, if $\Phi_{x,r}(h) = o(h)$ for a certain value of r, this relation remains true for any smaller r; taking r sufficiently near to 1 we obtain k as large as we please. Finally, it is

[1]) See Hardy and Littlewood [16] (for the case $r = k = 2$), Carleman [2], Sutton [1].

sufficient to prove (3) for the modified partial sums s_ν^* (§ 2.3); for $|s_\nu - f|^k \leqslant (|s_\nu^* - f| + |s_\nu - s_\nu^*|)^k$; hence, applying Jensen's inequality (§ 4.14), we obtain that $|s_\nu - f|^k \leqslant 2^{k-1}(|s_\nu^* - f|^k + |s_\nu - s_\nu^*|^k)$ and it is enough to observe that $|s_\nu - s_\nu^*|^k$ tends uniformly to 0.

Now, if $0 < \nu \leqslant n$,

$$s_\nu^*(x) - f(x) = \frac{1}{\pi}\int_0^\pi \varphi_x(t)\frac{\sin \nu t}{2\operatorname{tg}\frac{1}{2}t}\,dt = \frac{1}{\pi}\left(\int_0^{1/n} + \int_{1/n}^{\pi}\right) = \alpha_\nu^{(n)} + \beta_\nu^{(n)},$$

$$\left\{\frac{1}{n+1}\sum_{\nu=0}^{n}|s_\nu^* - f|^k\right\}^{1/k} \leqslant \left\{\frac{1}{n+1}\sum_{\nu=0}^{n}|\alpha_\nu^{(n)}|^k\right\}^{1/k} + \left\{\frac{1}{n+1}\sum_{\nu=0}^{n}|\beta_\nu^{(n)}|^k\right\}^{1/k},$$

and (i) will be established when we have shown that each of the terms on the right in the last inequality tends to 0 as $n \to \infty$. Since $|\sin \nu t/2\operatorname{tg}\frac{1}{2}t| < \nu$ for $0 < t \leqslant \pi$, we obtain that $|\alpha_\nu^{(n)}|$ does not exceed $\pi^{-1}\nu\,\Phi_{x,1}(1/n) \leqslant \nu\,\Phi_{x,1}(1/\nu) = \eta_\nu$. The relation $\Phi_{x,r}(h) = o(h)$ implies $\Phi_{x,1}(h) = o(h)$. Hence $\eta_\nu \to 0$ and

$$\text{(4)} \qquad \left\{\frac{1}{n+1}\sum_{\nu=0}^{n}|\alpha_\nu^{(n)}|^k\right\}^{1/k} \leqslant \left\{\frac{1}{n+1}\sum_{\nu=1}^{n}\eta_\nu^k\right\}^{1/k} \to 0.$$

Now observe that the β's are Fourier coefficients of the function equal to $\varphi_x(t)\,\frac{1}{2}\operatorname{ctg}\frac{1}{2}t$ for $1/n \leqslant t \leqslant \pi$, and to 0 for $-\pi \leqslant t < 1/n$. Applying the Hausdorff-Young inequality (§ 9.9.2) and supposing, as we may, that $r \leqslant 2$, we have

$$\text{(5)} \qquad \left\{\frac{1}{n+1}\sum_{\nu=0}^{n}|\beta_\nu^{(n)}|^k\right\}^{1/k} \leqslant \frac{1}{(n+1)^{1/k}}\left(\frac{1}{\pi}\int_{1/n}^{\pi}\left|\frac{\varphi_x(t)}{2\operatorname{tg}\frac{1}{2}t}\right|^r dt\right)^{1/r},$$

where $k = r'$. Replacing $2\operatorname{tg}\frac{1}{2}t$ by t, and integrating by parts, we see that the right-hand side of (5) does not exceed

$$\frac{1}{(n+1)^{1/k}}\left\{\left[\frac{\Phi_{x,r}(t)}{t^r}\right]_{1/n}^{\pi} + r\int_{1/n}^{\pi}\frac{\Phi_{x,r}(t)}{t^{r+1}}\,dt\right\}^{1/r} =$$

$$= \frac{1}{(n+1)^{1/k}}\left\{o(n^{r-1}) + \int_{1/n}^{\pi} o(t^{-r})\,dt\right\}^{1/r} =$$

$$= (n+1)^{-1/k}[o(n^{r-1}) + o(n^{r-1})]^{1/r} = o(1).$$

Hence the left-hand side of (5) tends to 0 and this, together with (4), proves (i).

The reader has no doubt noticed a curious feature of the above argument, namely, the less we suppose about the function, i. e. the smaller the number $r > 1$ is, the larger value for k we obtain. The argument however breaks down for $r = 1$ and the problem whether (3) is true for integrable functions remains unsolved, even when $k = 1$.

It is also of some interest to observe that it is sufficient to consider the values of r of the form $2l/(2l-1)$, $l = 1, 2, 3, \dots$, in which case the proof of the Hausdorff-Young theorem is simple (§ 9.12).

If $f \in L^r$, $r > 1$, the proof of (ii) is essentially the same as that of (i). We need only observe that, if $a \leqslant x \leqslant b$, then $\Phi_{x,r}(h) = o(h)$, $\Phi_{x,1}(h) = o(h)$ uniformly in x, and that the estimates we obtain are also uniform in x. If $f \in L$, we can find an interval (a_1, b_1), $a_1 < a \leqslant b < b_1$ such that f is bounded in (a_1, b_1). Let $f(x) = f'(x) + f''(x)$, where $f'(x) = f(x)$ in (a_1, b_1) and $f'(x) = 0$ elsewhere. If s'_ν and s''_ν denote the partial sums of $\mathfrak{S}[f']$ and $\mathfrak{S}[f'']$, then $s_\nu = s'_\nu + s''_\nu$ and

$$\left\{\frac{1}{n+1}\sum_{\nu=0}^{n}|s_\nu - f|^k\right\}^{1/k} \leqslant \left\{\frac{1}{n+1}\sum_{\nu=0}^{n}|s'_\nu - f'|^k\right\}^{1/k} + \left\{\frac{1}{n+1}\sum_{\nu=0}^{n}|s''_\nu - f''|^k\right\}^{1/k}.$$

The first term on the right tends to 0 uniformly in x, $a \leqslant x \leqslant b$, since f' is bounded and so belongs to every L^r. Since $f''(x) = 0$ for $a_1 < x < b_1$, the expression $|s''_\nu - f''|^k$ tends uniformly to 0 for $a \leqslant x \leqslant b$. Hence the second term on the right in the last inequality tends uniformly to 0 for $a \leqslant x \leqslant b$, and the proof of (ii) is complete.

We add that (3) is true if f is integrable and is continuous at the point x. This is a special case of (ii) when the interval (a, b) reduces to one point. The result holds if f has a simple discontinuity at x and if $2f(x) = f(x+0) + f(x-0)$.

10.11. When $r = k = 2$, Theorem 10.1(i) may be proved by a different argument which also works for general orthogonal systems of functions.

Let $\varphi_0(x), \varphi_1(x), \dots$ be a system of functions orthogonal and normal in an interval (a, b). If Σc_k^2 converges, and if the series $c_0\varphi_0(x) + c_1\varphi_1(x) + \dots$, with partial sums $s_n(x)$, is summable $(C, 1)$ in a set E, $|E| > 0$, to a function $s(x)$, then

$$\frac{1}{n+1}\sum_{k=0}^{n}[s_k(x)-s(x)]^2 \to 0 \quad \textit{as } n\to\infty,$$

for almost every $x \in E$[1]).

Let $\sigma_n(x)$ be the first arithmetic means of $\{s_n(x)\}$. We shall prove the following lemma: *If* $\Sigma c_n^2 < \infty$, *the series* $\Sigma[s_n(x)-\sigma_n(x)]^2/n$ *converges for almost every* $x \in (a, b)$. In view of Theorem 4.2(ii), it is sufficient to show that the latter series, integrated term by term over (a, b), is convergent. But

$$\sum_{n=1}^{\infty}\frac{1}{n}\int_a^b(s_n-\sigma_n)^2\,dx = \sum_{n=1}^{\infty}\frac{1}{n(n+1)^2}\sum_{k=1}^{n}k^2c_k^2 =$$

$$= \sum_{k=1}^{\infty}k^2c_k^2\sum_{n=k}^{\infty}\frac{1}{n(n+1)^2} \leqslant \sum_{k=1}^{\infty}k^2c_k^2\cdot\frac{1}{k^2} = \sum_{k=1}^{\infty}c_k^2,$$

and the lemma follows. Observing that, for every convergent series Σu_n, we have $u_1+2u_2+\ldots+nu_n = o(n)$ (§ 3.13(1)), we obtain that $(s_1-\sigma_1)^2+(s_2-\sigma_2)^2+\ldots+(s_n-\sigma_n)^2 = o(n)$ for almost every x. Now

$$\left\{\frac{1}{n+1}\sum_{k=0}^{n}(s_k-s)^2\right\}^{1/2} \leqslant \left\{\frac{1}{n+1}\sum_{k=0}^{n}(s_k-\sigma_k)^2\right\}^{1/2} + \left\{\frac{1}{n+1}\sum_{k=0}^{n}(\sigma_k-s)^2\right\}^{1/2},$$

and since of the two terms on the right the first is $o(1)$ for almost every x, and the second for every $x \in E$, the theorem is established.

10.2. In this paragraph we shall prove a number of theorems on the Abel and Cesàro means of Fourier series. The results will mostly bear on the behaviour of Fourier series in the whole interval $(0, 2\pi)$ and not at individual points.

10.21. An inequality for integrals. Let $f(x)$ be a non-negative function defined in an interval $(0, a)$, where for simplicity we suppose that $a < \infty$, and let $f^*(x)$, $0 < x \leqslant a$, be the function equimeasurable with f and non-increasing (§ 9.42). We put

(1) $$\theta(x; f) = \operatorname*{Sup}_{\xi}\frac{1}{x-\xi}\int_{\xi}^{x}f(t)\,dt, \quad 0 \leqslant \xi < x,\ 0 < x \leqslant a,$$

and similarly define $\theta(x; f^*)$. It is easy to see that for non-in-

---

[1]) Borgen [1], Zygmund [10].

creasing f, and in particular for f^*, the upper bound in (1) is attained when $\xi = 0$. The following theorem has important applications.

For any non-decreasing and non-negative function $s(t)$, $t \geqslant 0$,

$$(2) \qquad \int_0^a s\{\theta(x; f)\}\, dx \leqslant \int_0^a s\{\theta(x; f^*)\}\, dx\,^{1)}.$$

Given a non-negative function $g(x) \in L(0, a)$, let $e(y) = |E(y)|$, where $E(y)$ is the set of points x for which $g(x) > y$; then

$$(3) \qquad \int_0^a g(x)\, dx = -\int_0^{+\infty} y\, de(y) = \int_0^{+\infty} e(y)\, dy,$$

the second integral being a Riemann-Stieltjes integral. When g is bounded, the first equation follows at once if we observe that the approximate Lebesgue sums for the first integral are approximate Riemann-Stieltjes sums for the second integral. To obtain the result in the general case we apply the formula to the function $g_n(x) = \operatorname{Max}\{g(x), n\}$ and then make $n \to \infty$. The equality of the second and third integral follows by an integration by parts if we notice that $ye(y) \to 0$ as $y \to \infty$. This last relation is, in turn, a consequence of the fact that $ye(y)$ does not exceed the integral of $g(x)$ extended over the set of x for which $g(x) > y$.

Let $E(y_0)$ and $E^*(y_0)$ denote the sets of points where $\theta(x; f) > y_0$ and $\theta(x; f^*) > y_0$ respectively. Comparing the extreme terms of (3) we see that (2) will be established if we show that $|E(y_0)| \leqslant |E^*(y_0)|$ for every y_0. We break up the proof of this inequality into three stages.

(a) Given a continuous function $F(x)$, $0 \leqslant x \leqslant a$, let H denote the set of points x for each of which there is a point ξ, $0 \leqslant \xi < x$, such that $F(\xi) < F(x)$. Then H is an open set and is a sum of an at most enumerable system of open and non-overlapping intervals (α_k, β_k) such that $F(\alpha_k) \leqslant F(\beta_k)$ (it can easily be shown that actually we have $F(\alpha_k) = F(\beta_k)$, but this will not be required).

That H is open follows from the fact that the inequality $F(\xi) < F(x)$ is not impaired by slight changes of x. Let (α_k, β_k) be any of the open and non-overlapping intervals whose sum is E. Sup-

---

1) Hardy and Littlewood [17]; F. Riesz [7].

pose that $F(\alpha_k) > F(\beta_k)$, and let x_0 be the least number belonging to (α_k, β_k) and such that $F(x_0) = \frac{1}{2}[F(\alpha_k) + F(\beta_k)]$. No point ξ corresponding to x_0 can belong to (α_k, x_0), for the points x of this interval satisfy the inequality $F(x) \geqslant F(x_0)$. Hence $\xi < \alpha_k$, and the inequalities $F(\xi) < F(x_0)$, $F(x_0) < F(\alpha_k)$ give $F(\xi) < F(\alpha_k)$. Here we have a contradiction since the last inequality and the inequality $\xi < \alpha_k$ imply that $\alpha_k \in H$, which is false.

(b) If E is an arbitrary set contained in $(0, a)$, $|E| > 0$, then $\int\limits_E f\,dx \leqslant \int\limits_0^{|E|} f^*\,dx$.

This is a special case of a more general result established in § 9.42. An independent proof runs as follows. Let $f_1(x)$ be the function which is equal to $f(x)$ in E and to 0 elsewhere. Since $f_1(x) \leqslant f(x)$, we have $f_1^*(x) \leqslant f^*(x)$ and

$$\int\limits_E f\,dx = \int\limits_0^a f_1\,dx = \int\limits_0^a f_1^*\,dx = \int\limits_0^{|E|} f_1^*\,dx \leqslant \int\limits_0^{|E|} f^*\,dx.$$

(c) Let $E_1^*(y_0)$ denote the set of points where $\theta(x; f^*) \geqslant y_0$. Having fixed y_0 we shall write E, E^*, E_1^* instead of $E(y_0)$, $E^*(y_0)$, $E_1^*(y_0)$. If we put $F(x) = \int\limits_0^x f\,dt - y_0 x$, the set E becomes the set H of (a). If $\{(\alpha_k, \beta_k)\}$ is the sequence of open and non-overlapping intervals of which E consists, then, using the results obtained in (a) and (b),

$$\int\limits_{\alpha_k}^{\beta_k} f\,dx \geqslant y_0(\beta_k - \alpha_k), \qquad \int\limits_E f\,dx \geqslant y_0|E|, \qquad \int\limits_0^{|E|} f^*\,dx \geqslant y_0|E|.$$

Now $\theta(x; f^*) = \frac{1}{x}\int\limits_0^x f^*\,dt$; since the right-hand side of this equation is a non-increasing function of x, $|E_1^*|$ may be defined as the largest number x satisfying the inequality $\frac{1}{x}\int\limits_0^x f^*\,dt \geqslant y_0$. From this and the preceding inequality we infer that $|E| \leqslant |E_1^*|$. Therefore, if $\varepsilon > 0$, we have $|E(y_0 + \varepsilon)| \leqslant |E_1^*(y_0 + \varepsilon)|$ and, making $\varepsilon \to 0$, we obtain $|E(y_0)| \leqslant |E^*(y_0)|$. This completes the proof of (2).

10.211. We shall change the notation slightly. The function which we denoted by $\theta(x; f)$ will now be written $\theta_1(x; f)$. By $\theta_2(x; f)$ we shall denote $\operatorname{Sup}_{\xi} \frac{1}{\xi - x} \int_x^{\xi} f\, dt$ for $x < \xi \leqslant a$. If f_* denotes the function equimeasurable with f and non-decreasing, then

$$\int_0^a s\{\theta_1(x; f)\}\, dx \leqslant \int_0^a s\{\theta_1(x; f^*)\}\, dx, \qquad \int_0^a s\{\theta_2(x; f)\} \leqslant \int_0^a s\{\theta_2(x; f_*)\}.$$

The second inequality follows from the first by a simple transformation of the variable x. Let $\theta = \operatorname{Max}(\theta_1, \theta_2)$. It is not difficult to see that the inequality 10.21(2) holds for the new function θ if we introduce the factor 2 into the right-hand side. For $s(\theta) = \operatorname{Max}\{s(\theta_1), s(\theta_2)\} \leqslant s(\theta_1) + s(\theta_2)$ and so

$$\int_0^a s\{\theta(x; f)\}\, dx \leqslant \int_0^a s\{\theta_1(x; f^*)\}\, dx + \int_0^a s\{\theta_2(x; f_*)\}\, dx = 2\int_0^a s\{\theta(x; f^*)\}\, dx.$$

Thence, by a change of variable, we obtain

If (a, b) is a finite interval and

$$\theta(x; f) = \theta(x; f, a, b) = \operatorname{Sup}_{\xi} \frac{1}{x - \xi} \int_{\xi}^{x} f(t)\, dt, \qquad a \leqslant \xi \leqslant b,$$

then

$$\int_a^b s\{\theta(x; f)\}\, dx \leqslant 2\int_a^b s\left\{\frac{1}{x - a}\int_a^x f^*(t)\, dt\right\} dx,$$

where $f^*(x)$ is the function equimeasurable with $f(x)$ and non-decreasing.

10.22. Theorems of Hardy and Littlewood[1].

(i) *If $f \in L^r(a, b)$, $r > 1$, then $\theta(x; |f|) \in L^r(a, b)$ and*

$$\int_a^b \theta^r(x; |f|)\, dx \leqslant 2\left(\frac{r}{r-1}\right)^r \int_a^b |f|^r\, dx. \tag{1}$$

This follows from the remarks made in the previous section and from Theorem 4.17.

---

[1]) Hardy and Littlewood [17]; see also Paley [6].

The example of the function $f(x) = 1/x \log^2 x$ considered in the interval $(0, a)$, $0 < a < 1$, shows that, if $f \in L$, the function $\theta(x; |f|)$ need not be integrable.

(ii) *If $f \in L(a, b)$, then $\theta(x; |f|) \in L^\alpha(a, b)$ for every $0 < \alpha < 1$, and*

$$(2) \qquad \left\{\int_a^b \theta^\alpha(x; |f|)\, dx\right\}^{1/\alpha} \leqslant A_\alpha \int_a^b |f|\, dx,$$

where A_α depends on α and $b - a$ only.

(iii) *If $f \log^+ |f| \in L(a, b)$, then $\theta(x; |f|) \in L(a, b)$ and*

$$(3) \qquad \int_a^b \theta(x; |f|)\, dx \leqslant B \int_a^b |f| \log^+ |f|\, dx + C,$$

where B and C depend on $b - a$ only.

It is sufficient to prove (ii) and (iii) in the case of functions which are non-negative and non-increasing. We may also suppose that the interval (a, b) is of the form $(0, a)$. Then, applying Hölder's inequality,

$$\int_0^a \theta^\alpha(x; f)\, dx = \int_0^a \frac{dx}{x^{\alpha(1-\alpha)}} \left\{\frac{1}{x^\alpha}\int_0^x f\, dt\right\}^\alpha \leqslant \left\{\int_0^a \frac{dx}{x^\alpha}\right\}^{1-\alpha} \left\{\int_0^a \frac{dx}{x^\alpha}\int_0^x f\, dt\right\}^\alpha =$$

$$= \frac{a^{(1-\alpha)^2}}{(1-\alpha)^{1-\alpha}} \left\{\int_0^a f\, dt \int_t^a \frac{dx}{x^\alpha}\right\}^\alpha \leqslant \frac{a^{1-\alpha}}{1-\alpha}\left\{\int_0^a f\, dt\right\}^\alpha,$$

so that in the general case we have (2) with $A_\alpha^\alpha = 2a^{(1-\alpha)}/(1-\alpha)$.

To prove (3), let $I = \int_0^a f\, dx$, $J = \int_0^a f \log^+ f\, dx$; we shall denote by $B_1, B_2, \ldots$ constants which depend on a only. If f is non-negative and non-increasing, the left-hand side of (3) is equal to

$$(4) \qquad \int_0^a \frac{1}{x}\int_0^x f\, dt = \int_0^a f \log\frac{a}{x} dx \leqslant I \log^+ a + \int_0^a f \log^+ \frac{1}{x}\, dx.$$

Observing that $f \leqslant \text{Max}(e, f\log^+ f) \leqslant e + f\log^+ f$, we find that $I \leqslant J + ae = J + B_1$. On the other hand, since the monotonic functions $\Phi(x) = (x+1)\log(x+1) - x \leqslant (x+1)\log(x+1)$ and $\Psi(y) = e^y - y - 1 < e^y$ are complementary functions in the sense of Young (§ 4.11), an application of Young's inequality gives

$$\int_0^a 2f \cdot \tfrac{1}{2}\log^+ \frac{1}{x}\, dx < \int_0^a (2f+1)\log(2f+1)\, dx + \int_0^a e^{1/2 \log^+ 1/x}\, dx.$$

Since $2f+1 \leqslant \text{Max}(3, 3f)$, the first integral on the right is less than $B_2 J + B_3$. Collecting the results, we see that the left-hand side of (4) does not exceed $BJ + C$, and (3) is established.

Suppose now that $f(x)$ is of period 2π and integrable over $(0, 2\pi)$. Let

$$M(x; f) = \sup_{0<|t|\leqslant\pi} \frac{1}{t}\int_0^t |f(x+u)|\,du = \sup_{0<|t|\leqslant\pi} \frac{1}{t}\int_x^{x+t} |f(u)|\,du,$$

for $-\pi \leqslant x \leqslant \pi$. If we replace the condition $0<|t|\leqslant\pi$ by $-2\pi - x \leqslant t \leqslant 2\pi - x$, we increase the upper bound and we obtain, instead of $M(x; f)$, the function $\theta(x; |f|)$ formed for the interval $(-2\pi, 2\pi)$, and so

$$\int_{-\pi}^{\pi} s\{M(x; f)\}\,dx \leqslant \int_{-2\pi}^{2\pi} s\{\theta(x; |f|, -2\pi, 2\pi)\}\,dx.$$

Thence we easily obtain that

(iv) *The inequality* (1) *remains true if we replace the interval of integration* (a, b) *by* $(-\pi, \pi)$, *the function* $\theta(x; |f|)$ *by* $M(x; f)$, *and the factor* 2 *on the right by* 4.

(v) *The inequalities* (2) *and* (3) *hold if* (a, b) *is replaced by* $(-\pi, \pi)$, *and* $\theta(x; |f|)$ *by* $M(x; f)$. *The constant* A_α *will now depend only on* α, *and* B *and* C *will be absolute constants.*

Applications of the previous results to the theory of Fourier series are based on the following lemma.

(vi) *Let* $\chi(t, p)$, $-\pi \leqslant t \leqslant \pi$, *be a non-negative function depending on a parameter* p *and satysfying the conditions*

$$\text{(5a)} \quad \int_{-\pi}^{\pi} \chi(t, p)\,dt \leqslant K, \qquad \text{(5b)} \quad \int_{-\pi}^{\pi} \left| t\frac{\partial}{\partial t}\chi(t, p)\right| dt \leqslant K_1,$$

where K *and* K_1 *are independent of* p. *If*

$$h(x, p) = \int_{-\pi}^{\pi} f(x+t)\,\chi(t, p)\,dt,$$

then $\sup_p |h(x, p)| \leqslant AM(x; f)$, *where the constant* A *is independent of* f.

For let $F_x(t)$ be the integral of $|f(x+u)|$ over the interval $0 \leqslant u \leqslant t$ or $t \leqslant u \leqslant 0$. Integrating the formula defining $h(x, p)$ by parts and observing that $|F_x(t)| \leqslant |t|\, M(x; f)$, we find

$$|h(x,p)| \leqslant M(x;f)\left\{\int_{-\pi}^{\pi} \left| t\frac{\partial}{\partial t}\chi(t, p)\right| dt + \pi[\chi(\pi,p)+\chi(-\pi,p)]\right\}.$$

Integrating the integral of (5a) by parts and taking into account (5b), we see that $\pi[\chi(\pi, p)+\chi(-\pi, p)] \leqslant K + K_1$. Hence $|h(x,p)| \leqslant (2K_1 + K)\, M(x; f)$ and the lemma is established.

It is useful to observe that, if $t\,\partial\chi/\partial t$ is of constant sign, and if $\chi(\pm\pi, p)$ are bounded functions of p, then the inequality (5b) is a consequence of (5a). This follows at once if we drop the sign of absolute value in (5b) and integrate by parts.

If for $\chi(t, p)$ we take the Poisson kernel $P_r(t)$, the inequality (5a) is satisfied; also (5b) is true, for $t\,dP_r(t)/dt \leqslant 0$ and $P_r(\pm\pi) = O(1)$. Therefore,

(vii) *If $N(x; f)$ is the upper bound of $|f(r, x)|$ for $0 \leqslant r < 1$, where $f(r, x)$ denotes the Poisson integral of an integrable function $f(x)$, then $N(x; f) \leqslant AM(x; f)$, where A is an absolute constant.*

From this and from (iv) and (v) we obtain:

(viii) *The function $N(x; f)$ satisfies the inequalities*

$$\int_{-\pi}^{\pi} N^r(x; f)\, dx \leqslant A_r \int_{-\pi}^{\pi} |f|^r dx, \quad r > 1,$$

(6) $$\int_{-\pi}^{\pi} N^\alpha(x; f)\, dx \leqslant A_\alpha \left(\int_{-\pi}^{\pi} |f|\, dx\right)^{\alpha}, \quad 0 < \alpha < 1,$$

$$\int_{-\pi}^{\pi} N(x; f)\, dx \leqslant B \int_{-\pi}^{\pi} |f| \log^+ |f|\, dx + C,$$

where A_r depends only on r, A_α only on α, and B and C are absolute constants.

The Fejér kernel $K_n(t)$ satisfies (5a) but, as can easily be shown, not (5b). The same may be said of the kernel $K_n^\delta(t)$, $0 < \delta < 1$, which, besides, is not of constant sign. The kernel $K_n^\delta(t)$, $0 < \delta \leqslant 1$, can however be majorised by a function which satisfies the inequalities (5). For

$$(7)\qquad |K_n^{\delta}(t)| \leqslant L_n^{\delta}(t) = \frac{c(\delta)\,n}{1+(n\,|t|)^{\delta+1}}, \quad 0<\delta\leqslant 1, \quad n\geqslant 1, \quad |t|\leqslant\pi,$$

where $c(\delta)$ depends on δ only. To prove this inequality, which is due substantially to Fejér [1]), it is sufficient to observe that $L_n^{\delta}(t) \geqslant \frac{1}{2}c(\delta)\,n$ for $n|t|\leqslant 1$, $L_n^{\delta}(t) \geqslant \frac{1}{2}c(\delta)/n^{\delta}|t|^{\delta+1}$ for $n|t|\geqslant 1$, and to take into account the inequalities 3.3(2). The reader will verify that the function $\chi(t,p) = L_n^{\delta}(t)$ satisfies (5a); that the inequality (5b) is also satisfied follows from the fact that $t\cdot dL_n^{\delta}(t)/dt \leqslant 0$ and that $L_n^{\delta}(\pm\pi) = O(1)$.

Let $\sigma_n^{\delta}(x;f)$ be the Cesàro means of order δ for $\mathfrak{S}[f]$. Observing that $|\sigma_n^{\delta}(x;f)| \leqslant \frac{1}{\pi}\int_{-\pi}^{\pi} |f(x+t)|\,L_n^{\delta}(t)\,dt$, we obtain:

(ix) *If* $N_{\delta}(x;f)$, $0<\delta\leqslant 1$, *is the upper bound of* $|\sigma_n^{\delta}(x;f)|$ *for* $1\leqslant n<\infty$, *then* $N_{\delta}(x;f)\leqslant AM(x;f)$ *with* A *depending only on* δ; *the function* $N_{\delta}(x;f)$ *satisfies inequalities similar to* (6), *where the constants* A_r, A_α, B, C *will now depend also on* δ [2]).

The theorem remains true for $\delta>1$. This follows from the fact, which is easy to verify (§ 3.13), that $N_{\delta}(x;f)$ is a non-increasing function of δ.

We return to the case of harmonic functions $f(r,x)$. If $0\leqslant\varphi<\frac{1}{2}\pi$, we denote by $S_\varphi(x)$ the part of the unit circle limited by two chords through e^{ix} at angles α to the radius, and the perpendiculars upon them from the origin. Let $N(x;f,\varphi)$ be the upper bound of $|f(r,\theta)|$ for $z=re^{i\theta}$, $r<1$, belonging to $S_\varphi(x)$.

(x) *There is a number* A *depending only on* φ *such that* $N(x;f,\varphi)\leqslant AM(x;f)$. *The function* $N(x;f,\varphi)$ *satisfies inequalities similar to* (6), *except that the constants* A_r, A_α, B, C *will now depend also on* φ.

It is only the first part of this theorem which needs a proof. If $z=re^{i\theta}$, $r<1$, is any point belonging to $S_\varphi(x)$, and $\zeta=re^{i(\theta-x)}$, then

---

[1]) Fejér [10]. If we replace n by $n+1$ in the numerator of the last ratio, the inequality will hold for $n\geqslant 0$.

[2]) The theorem remains true if in the definition of $N_{\delta}(x;f)$ we suppose that n runs from 0 to ∞. It suffices to modify the definition of $L_n^{\delta}(t)$ slightly (see the preceding footnote).

$$f(r, \theta) = \int_{-\pi}^{\pi} f(x+t)\, \chi(t, \zeta)\, dt, \text{ where } \chi(t, \zeta) = \frac{1}{\pi} P_r(t + x - \theta).$$

The expression $\chi(t, \zeta)$ here depends on the variable t and the parameter ζ belonging to the region $S_\varphi(0)$. That the inequality (5a) is satisfied, is apparent. The left-hand side of (5b) takes the form $\frac{1}{\pi}\int_{-\pi}^{\pi} \left| t \frac{d}{dt} P_r(t+\xi) \right| dt$, where $\xi = x - \theta$. Supposing, to fix ideas, that $\xi > 0$, we break up the interval of integration into three parts $(-\pi, -\xi)$, $(-\xi, 0)$, $(0, \pi)$, in each of which the expression under the sign of absolute value is of constant sign. Integrating by parts, and observing that $P_r(0) = O(1/(1-r))$, $\xi = O(1-r)$, we obtain the desired inequality.

Proposition (x), suitably modified, can be extended to general classes $H^p, p > 0$ (§ 7.51):

(xi) *If $F(z)$ is a function regular for $|z| < 1$, and if $\mu_p(r; F) \leqslant \lambda^p$, $0 \leqslant r < 1$, $p > 0$, then $\mathfrak{M}_p^p[N(x; F, \varphi)] \leqslant A_\varphi \lambda^p$, where A_φ depends on φ only.*

This theorem is a consequence of (x) if $p = 2$. In the general case we have $F(z) = G(z) B(z)$, where $|B(z)| \leqslant 1$, $G(z)$ is regular and non-vanishing, and $\mu_p(r; G) \leqslant \lambda^p$ (§ 7.53(v)). The function $G^{p/2}(z)$ is regular and belongs to H^2. Since $\mu_2(r; G^{p/2}) = \mu_p(r; G) \leqslant \lambda^p$, we obtain $\mathfrak{M}_2^2[N(x; G^{p/2}, \varphi)] \leqslant A_\varphi \lambda^p$, and it is sufficient to observe that the left-hand side of the last inequality is equal to the expression $\mathfrak{M}_p^p[N(x; G, \varphi)] \geqslant \mathfrak{M}_p^p[N(x; F, \varphi)]$.

The most important special case of (xi) is when $\varphi = 0$ and A_φ reduces to a radius of the circle.

The theorems established in this section elucidate certain results of Chapter IV. To prove, for example, that, if $f \in L^r$, $r > 1$, then $\mathfrak{M}_r[f - \sigma_n] \to 0$ (§ 4.35), it is sufficient to observe that $|f(x) - \sigma_n(x)|^r$ tends almost everywhere to 0 and is dominated by an integrable function. Similarly Theorem 7.56(iii) is an easy consequence of (xi).

10.23. We conclude this paragraph by a few remarks on the function $\widetilde{f}(x) = \operatorname{Sup}_h |f_h(x)|$, where

$$\bar{f}_h(x) = -\frac{1}{\pi}\int_h^\pi \frac{f(x+t)-f(x-t)}{2\operatorname{tg}\frac{1}{2}t}\,dt, \quad 0 < h < \pi,$$

f denoting an integrable and periodic function. In § 7.11 we showed that $\widetilde{f}(x)$ is finite almost everywhere. Completing the results of §§ 7.21, 7.24, we shall show that

$$\mathfrak{M}_r[\widetilde{f}] \leqslant A_r\, \mathfrak{M}_r[f], \;\; r > 1; \quad \mathfrak{M}_\alpha[\widetilde{f}] \leqslant A_\alpha\, \mathfrak{M}\,[f], \quad 0 < \alpha < 1;$$

$$\mathfrak{M}\,[\widetilde{f}] \leqslant B\, \mathfrak{M}\,[f \log^+ |f|] + C,$$

where A_r depends only on r, A_α only on α, and B and C are absolute constants. It is sufficient to prove the first of these inequalities only, the proof of the remaining being similar. Let us put $\psi_x(t) = f(x+t) - f(x-t)$; then

$$\bar{f}_{1-r}(x) - \bar{f}(r,x) = \frac{1}{\pi}\int_0^{1-r} \psi_x(t)\, Q_r(t)\, dt - \frac{1}{\pi}\int_{1-r}^{\pi} \psi_x(t)\, R_r(t)\, dt = G_r(x) + H_r(x),$$

where $Q_r(t) = r\sin t/(1-2r\cos t + r^2)$, $R_r(t)$ denotes the ratio $(1-r)^2/2\operatorname{tg}\frac{1}{2}t\cdot(1-2r\cos t + r^2)$, and $\bar{f}(r,x)$ is the harmonic function conjugate to $f(r,x)$. Since $|Q_r(t)| < 1/(1-r)$, we have $|G_r(x)| \leqslant M(x;f)$. Integrating by parts and observing that $t\,R_r(t) = O(1)$ for $t = 1-r$, and that $\int_{1-r}^{\pi} t\left|\frac{d}{dt}R_r(t)\right| dt = O(1)$, we find that $|H_r(x)|$, and so also $|\bar{f}_{1-r}(x) - \bar{f}(r,x)|$, does not exceed a multiple of $M(x;f)$.

Suppose now that $f \in L^r$, $r > 1$; then the function $\bar{f}(x) = \bar{f}(x;+0)$ belongs to L^r, and $\bar{f}(r,x)$ is the Poisson integral of $\bar{f}(x)$. Hence $|\bar{f}_{1-r}(x)| \leqslant |\bar{f}_{1-r}(x) - \bar{f}(r,x)| + |\bar{f}(r,x)| < D\{M(x;f) + M(x;\bar{f})\}$, where D is an absolute constant. This inequality gives $|\widetilde{f}(x)| \leqslant D\{M(x;f) + M(x;\bar{f})\}$, $\mathfrak{M}_r[\widetilde{f}] \leqslant D\{\mathfrak{M}_r[M(x;f)] + \mathfrak{M}_r[M(x;\bar{f})]\}$. In view of Theorem 10.22(iv), the right-hand side of the last inequality does not exceed a multiple of $\mathfrak{M}_r[f] + \mathfrak{M}_r[\bar{f}]$ and it suffices to apply Theorem 7.21.

10.3. Partial sums of $\mathfrak{S}[f]$ for $f \in L^2$. The theory of summability of Fourier series by Abel's method, or Cesàro's methods of positive order, is in a state which may be described as satisfactory. The situation is adequately represented when we say that what we need there most are problems, that is interesting problems. Achievements of the modern theory of real functions have left means at our disposal which seem to be sufficient to cope with problems of summability, although the latter may in some cases be fairly difficult.

The situation is different when we consider the behaviour of partial sums. Several results have been obtained for the convergence at individual points, but as regards convergence or divergence almost everywhere, our knowledge is still very scanty. Problems which suggest themselves to the beginner (for example the problem whether $\mathfrak{S}[f]$ must converge at one point at least when f is continuous) seem to be far from being solved. It is true that in the last few years a number of important results have been obtained, connected with the names of Kolmogoroff and Seliverstoff, Plessner, and Littlewood and Paley, but much more still remains to be done.

10.31. Theorems of Kolmogoroff[1]. Let $f(x)$ be a function of the class L^2 and let $s_n(x)$ be the partial sums of the Fourier series

$$(1) \qquad \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

of $f(x)$. Since $\mathfrak{M}_2[f - s_n] \to 0$, there is a subsequence $\{s_{n_k}(x)\}$ of $\{s_n(x)\}$ which converges almost everywhere to $f(x)$ (§ 4.2). We shall now prove that for $\{n_k\}$ we may take a sequence independent of f.

(i) *If $n_{k+1}/n_k > \lambda > 1$, $k = 1, 2, \dots$, the partial sums $s_{n_k}(x)$ of $\mathfrak{S}[f]$, $f \in L^2$, converge almost everywhere to $f(x)$.*

A series Σc_i is said to possess a gap (u, v) if $c_i = 0$ for $u < i \leqslant v$. We shall require the following lemma. *If a series Σc_i, with partial sums s_n, possesses infinitely many gaps (m_k, m'_k) such that $m'_k/m_k > \lambda > 1$, and is summable $(C, 1)$ to sum s, then s_{m_k}, and so also $s_{m'_k}$, converges to s.*

Let $s = 0$, $s_0 + s_1 + \dots + s_n = (n+1)\sigma_n$. Then

$$(2) \qquad \begin{aligned} (m'_k - m_k)\, s_{m_k} &= s_{m_k+1} + s_{m_k+2} + \dots + s_{m'_k} = \\ &= (m'_k + 1)\,\sigma_{m'_k} - (m_k + 1)\,\sigma_{m_k} = o(m'_k) + o(m_k) = o(m'_k), \end{aligned}$$

whence $s_{m_k} = o(1)$ and the lemma is established. In particular

[1] Kolmogoroff [8]; Marcinkiewicz [1].

(ii) *If the Fourier series of an integrable function $f(x)$ possesses infinitely many gaps (m_k, m'_k) such that $m'_k/m_k > \mu > 1$, the partial sums $s_{m_k}(x)$ converge almost everywhere to $f(x)$.*

Now, in order to prove (i), we split (1) into consecutive blocks of terms $n_k < n \leqslant n_{k+1}$, $n_0 = 0$, including $\frac{1}{2} a_0$ in the first block; we then break up the whole series into two, one consisting of blocks with even, the other with odd, indices. By the Riesz-Fischer theorem, these series are Fourier series of functions f' and f'' respectively. For each series the terms with indices n_k are either at the end of or immediately preceding a gap, and so, by (ii), the partial sums of the two series, viz. $s'_{n_k}(x)$ and $s''_{n_k}(x)$, converge almost everywhere. The same is true for $s_{n_k}(x) = s'_{n_k}(x) + s''_{n_k}(x)$.

(iii) *Let* $s(x) = \operatorname{Sup}_k |s_{n_k}(x)|$. *Under the hypothesis of* (i), $s(x)$ *belongs to* L^2 *and* $\mathfrak{M}_2[s] \leqslant A_\lambda \mathfrak{M}_2[f]$, *where* A_λ *depends on* λ *only.*

Denoting by $B_1, B_2, \ldots$ constants depending exclusively on λ, we obtain from (2) that $\operatorname{Sup} |s_{m_k}| \leqslant B_1 \operatorname{Sup} |\sigma_{m_k}|$. Hence, under the hypothesis of (ii), $\operatorname{Sup} |s_{m_k}(x)| \leqslant B_1 \operatorname{Sup} |\sigma_{m_k}(x)| \leqslant B_2 M(x; f)$ (§ 10.2(ix)). Therefore, if f', f'', s'_{n_k}, s''_{n_k} have the same meaning as before,

$$s(x) \leqslant \operatorname{Sup} |s'_{n_k}(x)| + \operatorname{Sup} |s''_{n_k}(x)| \leqslant B_2\{M(x; f') + M(x; f'')\},$$

$$\mathfrak{M}_2[s] \leqslant B_2\{\mathfrak{M}_2[M(x; f')] + \mathfrak{M}_2[M(x; f'')]\} \leqslant B_3\{\mathfrak{M}_2[f'] + \mathfrak{M}_2[f'']\},$$

and it is sufficient to observe that, in view of Parseval's relation, the last expression in curly brackets does not exceed the sum $\mathfrak{M}_2[f] + \mathfrak{M}_2[f] = 2\mathfrak{M}_2[f]$.

10.32. Convergence of a class of trigonometrical series [1]). An immediate consequence of Theorem 3.71 is that, if $\Sigma (a_n^2 + b_n^2) \log^2 n < \infty$, the series 10.31(1) converges almost everywhere. For from the last inequality and the Riesz-Fischer theorem we see that the trigonometrical series with coefficients $a_n \log n$, $b_n \log n$, is a Fourier series and, applying the first part

[1]) Kolmogoroff and Seliverstoff [1], [2], Plessner [4]. The method of the proof seems to have been used first by Jerosch and Weyl [1], to obtain much weaker results.

of Theorem 3.71 to it, we obtain the desired result. Now we shall prove a more general theorem.

(i) *If the series* $\sum_{n=2}^{\infty}(a_n^2+b_n^2)\log n$ *converges, the series* 10.31(1) *converges almost everywhere.*

The argument which we shall use to prove this theorem is not less interesting than the result itself, and may be used in many problems.

Without loss of generality we may suppose that $a_0=a_1=b_1=0$. Let $E_n(x)$ and $H_n(x)$, $n=0, 1, \dots$, denote the partial sums of the series

$$\sum_{n=2}^{\infty}\frac{\cos nx}{\sqrt{\log n}}, \qquad \sum_{n=2}^{\infty}\frac{\cos nx}{\log n}$$

respectively. Let $n(x)$, $0 \leqslant x \leqslant 2\pi$, be any measurable function taking non-negative integral values and bounded above by some integer N. If $s_\nu(x)$ are the partial sums of 10.31(1), and if the series $\sum_{2}^{\infty}(a_n\cos nx+b_n\sin nx)\sqrt{\log n}$ is $\mathfrak{S}[g]$, then

$$s_\nu(x)=\frac{1}{\pi}\int_0^{2\pi}g(t)\,E_\nu(t-x)\,dt.$$

Putting $\nu=n(x)$, integrating over the interval $(0, 2\pi)$, and using Schwarz's inequality, we obtain

$$\left|\int_0^{2\pi}s_{n(x)}(x)\,dx\right|=\left|\frac{1}{\pi}\int_0^{2\pi}dx\int_0^{2\pi}g(t)\,E_{n(x)}(t-x)\,dt\right|=$$

$$=\left|\frac{1}{\pi}\int_0^{2\pi}g(t)\,dt\int_0^{2\pi}E_{n(x)}(t-x)\,dx\right|\leqslant\mathfrak{M}_2[g]\,\mathfrak{M}_2\left[\frac{1}{\pi}\int_0^{2\pi}E_{n(x)}(t-x)\,dx\right].$$

The square of the last factor is equal to

$$\frac{1}{\pi}\int_0^{2\pi}dt\left[\int_0^{2\pi}E_{n(x)}(t-x)\,dx\right]\left[\int_0^{2\pi}E_{n(x')}(t-x')\,dx'\right]=$$

(1)

$$=\int_0^{2\pi}\int_0^{2\pi}dx\,dx'\left\{\frac{1}{\pi}\int_0^{2\pi}E_{n(x)}(x-t)\,E_{n(x')}(x'-t)\,dt\right\}.$$

The expression in curly brackets is equal to $H_m(x-x')$, where $m = m(x, x') = \text{Min}\{n(x), n(x')\}$, and so the right-hand side of (1) does not exceed

$$\int_0^{2\pi}\int_0^{2\pi} \{|H_{n(x)}(x-x')| + |H_{n(x')}(x-x')|\}\, dx\, dx' =$$

(2)

$$= 2\int_0^{2\pi}\int_0^{2\pi} |H_{n(x)}(x-x')|\, dx\, dx'.$$

In § 5.12 we saw that $\mathfrak{M}[H_\nu] = O(1)$. Hence, integrating first with respect to x' and then with respect to x, we see that the right-hand side of (2) is less than an absolute constant A, and

$$\text{(3)} \qquad \left|\int_0^{2\pi} s_{n(x)}(x)\, dx\right| \leqslant A\, \mathfrak{M}_2[g] = A\left\{\pi \sum_{\nu=2}^{\infty} (a_\nu^2 + b_\nu^2) \log \nu\right\}^{1/2}.$$

This is a fundamental inequality from which the theorem follows comparatively easily. For let $\varphi_N(x) = \text{Sup}\, s_n(x)$, $0 \leqslant n \leqslant N$, $\psi_N(x) = \text{Sup}\{-s_n(x)\}$, $0 \leqslant n \leqslant N$. Since $s_0(x) = 0$, the functions φ_N and ψ_N are non-negative. By choosing suitable functions $n(x)$, the inequality (3) gives $\mathfrak{M}[\varphi_N] \leqslant A\,\mathfrak{M}_2[g]$, $\mathfrak{M}[\psi_N] \leqslant A\,\mathfrak{M}_2[g]$. The sequences $\{\varphi_N(x)\}$ and $\{\psi_N(x)\}$ are non-decreasing and so, putting $\Phi(x) = \lim \varphi_N(x)$, $\Psi(x) = \lim \psi_N(x)$, we have $\mathfrak{M}[\Phi] \leqslant A\,\mathfrak{M}_2[g]$, $\mathfrak{M}[\Psi] \leqslant A\,\mathfrak{M}_2[g]$. The functions Φ and Ψ, being integrable, are finite almost everywhere and, since $\Phi(x) = \text{Sup}\, s_n(x)$, $\Psi(x) = \text{Sup}\{-s_n(x)\}$, the sequence $\{s_n(x)\}$ is bounded for almost every x.

If $\Omega(x)$ denotes the upper bound of $|s_m(x) - s_n(x)|$ for all values of m and n, then $\Omega(x) \leqslant \Phi(x) + \Psi(x)$, and so we have $\mathfrak{M}(\Omega) \leqslant 2A\,\mathfrak{M}_2[g]$.

To prove that $\{s_n(x)\}$ converges almost everywhere, let $\Omega_M(x) = \text{Sup}\,|s_n(x) - s_m(x)|$ for all possible values of $m \geqslant M$ and $n \geqslant M$, and let $g_M(x) \sim \sum_{M+1}^{\infty} (a_n \cos nx + b_n \sin nx)\sqrt{\log n}$. The function Ω_M is the Ω corresponding to g_M, so that $\mathfrak{M}[\Omega_M] \leqslant 2A\,\mathfrak{M}_2[g_M]$. In view of Parseval's formula, $\mathfrak{M}_2[g_M] \to 0$ as $M \to \infty$, and so we also have $\mathfrak{M}[\Omega_M] \to 0$. Since $\{\Omega_M\}$ is a non-increasing sequence, we conclude that $\mathfrak{M}[\lim \Omega_M] = 0$, i. e. $\lim \Omega_M(x) = 0$ for almost every x. In other words, the sequence $\{s_n(x)\}$ converges for almost every x, and (i) is established.

(ii) *If the series* 10.31(1) *belongs to* L^2, *the partial sums* $s_n(x)$ *of the series are* $o(\sqrt{\log n})$ *for almost every* x.

For if $a_n/\sqrt{\log n} = a'_n$, $b_n/\sqrt{\log n} = b'_n$, $n = 2, 3, \dots$, then $\Sigma (a'^2_n + b'^2_n) \log n < \infty$. Hence the series $\Sigma (a'_n \cos nx + b'_n \sin nx)$ converges for almost every x, and it is sufficient to prove the following lemma. *If* $0 < l_2 \leqslant l_3 \leqslant \dots \to \infty$, *and if the series* $u_2/l_2 + u_3/l_3 + \dots$ *converges, then* $u_2 + u_3 + \dots + u_n = o(l_n)$.

Let $s_n = u_2 + \dots + u_n$, $r_n = u_n/l_n + u_{n+1}/l_{n+1} + \dots$ Taking m such that $|r_k| < \varepsilon$ for $k > m$, and applying Abel's transformation, we have

$$s_n - s_m = \sum_{m+1}^{n} \frac{u_k}{l_k} l_k = r_{m+1} l_{m+1} + \sum_{m+2}^{n} r_k (l_k - l_{k-1}) - r_{n+1} l_n$$

for $n > m$. The last expression does not exceed $2\varepsilon l_n$ in absolute value. Hence $|s_n| \leqslant |s_n - s_m| + |s_m| \leqslant 2\varepsilon l_n + |s_m| < 3\varepsilon l_n$ if n is large enough. Since ε is arbitrary, the lemma is established.

(iii) *If the series* 10.31(1) *belongs to* L^2, *the series*

$$\sum_{n=2}^{n} \frac{a_n \cos nx + b_n \sin nx}{\sqrt{\log n}}$$

converges almost everywhere.

Since the sequence $\mu_n = 1/\sqrt{\log n}$, $n = 2, 3, \dots$ is convex, (iii) follows from (ii) and the lemma established in § 3.71.

10.33. The theorems of the previous sections have been extended by Littlewood and Paley to the case of functions belonging to L^r, $r > 1$. In this case the arguments are more difficult and require new devices. We shall state here, without proof, the most important of the Littlewood-Paley results [1]). Let $s_n(x)$ denote the partial sums of the series 10.31(1), which is the Fourier series of a function $f(x)$; then

(i) *If* $f \in L^r$, $r > 1$, *and if the sequence* $\{n_k\}$ *satisfies an inequality* $n_{k+1}/n_k > \lambda > 1$, $k = 1, 2, \dots$, *the sequence* $\{s_{n_k}(x)\}$ *converges to* $f(x)$ *for almost every* x; *the function* $\operatorname{Sup}_k |s_{n_k}(x)|$ *belongs to* L^r.

---

[1]) See Littlewood and Paley [1]. Detailed proofs have not yet been published, but some indications as to the methods of proofs will be found in Paley [1], where similar results are obtained for the orthogonal system defined in § 1.8.5.

(ii) *If* $f \in L^r$, $1 < r \leqslant 2$, *then, almost always,* $s_n(x) = o\,(\log n)^{1/r}$ *and the series*

$$\sum_{n=2}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{(\log n)^{1/r}}$$

converges.

(iii) *If* $\{\varepsilon_k\}$ *is any sequence of numbers of which each has one of the three values* $0, 1, -1$, *and if* $f \in L^r$, $r > 1$, $n_{k+1}/n_k > \lambda > 1$, *the series*

$$\sum_{k=1}^{\infty} \varepsilon_k \sum_{n=n_k+1}^{n_{k+1}} (a_n \cos nx + b_n \sin nx)$$

is the Fourier series of a function $g \in L^r$.

We add a few remarks.

Proposition (i) is false for $r = 1$; more precisely: *for any sequence* $\{\lambda_k\}$ *of positive numbers there is an integrable function* $f(x)$, *and a sequence* $\{n_k\}$ *such that* $n_{k+1}/n_k > \lambda_k$ *and that* $s_{n_k}(x)$ *diverges almost everywhere.* For the proof we refer the reader to Kolmogoroff [7]. Although the result is not stated there explicitely, it is an easy consequence of the argument used.

Theorem (ii) is established for $r \leqslant 2$ only, so that for functions $f \in L^s$, $s > 2$, and in particular for continuous functions, it does not give more than Theorems 10.32(ii), (iii). It is not excluded that proposition (ii) is false for $r > 2$.

The meaning of (iii) will be understood better, if the reader compares this result with the theorems established in § 5.6.

10.4. Summability C **of Fourier series.** In Chapter II we studied various tests ensuring the convergence of the Fourier series of a function $f(x)$ at a given point. All those tests represent sufficient conditions only, and the problem of finding a *necessary and sufficient* condition (which would not be a more or less disguised tautology) remains unsolved. The situation is the same when, instead of ordinary convergence, we consider summability by an assigned Cesàro mean, e. g. summability $(C, 1)$: Fejér's fundamental theorem (§ 3.21) gives a sufficient condition only. We therefore change the problem and ask not when $\mathfrak{S}[f]$ is summable by *some particular* mean, but when it is summable by *some mean or another*, i. e. when it is *summable C.* In this form the problem was first stated by Hardy and Littlewood, who also gave a complete solution. This solution has been precised at certain points by a number of writers, in particular by Bosanquet. A new approach to the problem was found by Plessner.

We begin by proving a number of auxiliary theorems which are interesting and important in themselves.

10.41. Suppose that $f(x)$ is defined in the neighbourhood of a point x_0 and that, for small values of $|t|$,

$$f(x_0+t)=\alpha_0+\frac{1}{1!}\alpha_1 t+\frac{1}{2!}\alpha_2 t^2+\dots+\frac{1}{(r-1)!}\alpha_{r-1}t^{r-1}+\frac{1}{r!}(\alpha_r+\varepsilon_t)t^r,$$

where the α's are constants and $\varepsilon_t=\varepsilon_{r,t}\to 0$ with t. The number α_s, $1\leqslant s\leqslant r$, will then be called the s-th *generalized derivative* of f at the point x_0. It is plain that, if $f^{(s)}(x_0)$, $s=1,2,\dots$, exists and is finite, then the s-th generalized derivative α_s exists and is equal to $f^{(s)}(x_0)$. For applications to the theory of trigonometrical series it is convenient to modify this definition and to consider the cases of even and odd suffixes separately. Let $\varphi_{x_0}(t)=\frac{1}{2}[f(x_0+t)+f(x_0-t)]$, $\psi_{x_0}(t)=\frac{1}{2}[f(x_0+t)-f(x_0-t)]$. If either

$$\varphi_{x_0}(t)=\beta_0+\frac{\beta_2}{2!}t^2+\dots+\frac{\beta_{2k-2}}{(2k-2)!}t^{2k-2}+(\beta_{2k}+\varepsilon_t)\frac{t^{2k}}{(2k)!},\quad\text{or}$$

$$\psi_{x_0}(t)=\beta_1 t+\frac{\beta_3}{3!}t^3+\dots+\frac{\beta_{2k-1}}{(2k-1)!}t^{2k-1}+(\beta_{2k+1}+\varepsilon_t)\frac{t^{2k+1}}{(2k+1)!},$$

where $\varepsilon_t\to 0$ as $t\to 0$, and the β's are constants, then β_j will be called the j-th *generalized symmetric derivative* of $f(x)$ at the point x_0, and will be denoted by $f_{(j)}(x_0)$[1]. The existence of $f_{(j)}(x_0)$ involves that of $f_{(j-2)}(x_0)$. The following theorem is a generalization of Theorem 3.5.

If $f_{(r)}(x_0)$ exists, the Fourier series of $f(x)$, differentiated term by term r times, is, at the point x_0, summable (C,α), $\alpha>r$, to the value $f_{(r)}(x_0)$[2].

We observe that, given $2s+1$ numbers $\xi_0,\xi_1,\dots,\xi_{2s}$, there is a trigonometrical polynomial $T(x)$ of order $\leqslant s$, such that $T^{(j)}(x_0)=\xi_j$, $0\leqslant j\leqslant 2s$. This is easily seen when we represent $T(x)$ in the complex form and write equations for the coefficients. Since the

---

[1]) The generalized derivatives were first introduced by de la Vallée-Poussin [4].

[2]) de la Vallée-Poussin [4], Gronwall [3], Young [4], Zygmund [15].

theorem is obvious in the case of trigonometrical polynomials, we may, by substracting a polynomial $T(x)$ from $f(x)$, suppose that $f_{(r)}(x_0)=f_{(r-2)}(x_0)=\ldots=0$. If $K_n^\alpha(t)$ denotes the (C,α) kernel, and $\sigma_n^\alpha(x)$ are the (C,α) means of $\mathfrak{S}[f]$, the (C,α) means of $\mathfrak{S}^{(r)}[f]$ are equal to $\{\sigma_n^\alpha(x)\}^{(r)}$, i. e. to

$$\frac{(-1)^r}{\pi}\int_{-\pi}^{\pi} f(t)\frac{d^r}{dt^r}K_n^\alpha(x-t)\,dt=\frac{2(-1)^r}{\pi}\int_0^\pi \tfrac{1}{2}[f(x+t)+(-1)^r f(x-t)]\frac{d^r}{dt^r}K_n^\alpha(t)\,dt.$$

In what follows, $C, C_1, C_2, \ldots$ will denote positive constants independent of the variables t and n. The proof of the theorem is an easy consequence of the following lemma:

If $0\leqslant r<\alpha$, *then* (i) $\int_0^\pi t^r\left|\frac{d^r}{dt^r}K_n^\alpha(t)\right|dt<C$, *and* (ii) *the expression* $\frac{d^r}{dt^r}K_n^\alpha(t)$ *tends uniformly to* 0 *in any interval* $0<\eta\leqslant t\leqslant\pi$.

Let us take this lemma for granted for the moment, and let $\delta>0$ be an arbitrary number. If $f_{(r)}(x_0)=f_{(r-2)}(x_0)=\ldots=0$, then, since $2/\pi r!<1$, the expression $|\{\sigma_n^\alpha(t)\}^{(r)}|$ does not exceed

$$\int_0^\pi|\varepsilon_t|\,t^r\left|\frac{d^r}{dt^r}K_n^\alpha(t)\right|dt=\int_0^\eta+\int_\eta^\pi=A+B,$$

where η is so chosen that $|\varepsilon_t|\leqslant\delta/2C$ for $0<t\leqslant\eta$. Then $|A|\leqslant C\cdot\delta/2C=\frac{1}{2}\delta$, and since, in view of (ii), $B\to0$ as $n\to\infty$, we obtain that $|\{\sigma_n^\alpha(x_0)\}^{(r)}|<\delta$ for $n\geqslant n_0$. Hence $\{\sigma_n^\alpha(x_0)\}^{(r)}\to0$ and the theorem is established.

Let $u(\beta,n,t)=\sum_{\nu=0}^{n}A_\nu^\beta e^{i\nu t}$. Abel's transformation shows that $u(\beta,n,t)=[-A_n^\beta e^{i(n+1)t}+u(\beta-1,n,t)]/(1-e^{it})$, and so

$$(1)\qquad u(\beta,n,t)=-e^{i(n+1)t}\sum_{j=1}^{s}\frac{A_n^{\beta-j+1}}{(1-e^{it})^j}+\frac{u(\beta-s,n,t)}{(1-e^{it})^s}.$$

To prove the lemma we use, besides (1), the relations 3.3(3) and the first formula in 3.11(1). Then

$$K_n^\alpha(t)=\frac{1}{2A_n^\alpha\sin\frac{1}{2}t}\,\mathfrak{I}\,[e^{i(n+1/2)t}\,u(\alpha-1,n,-t)]=$$

$$= \frac{1}{2A_n^\alpha \sin\frac{1}{2}t}\,\Im\left[-e^{-\frac{1}{2}it}\sum_{j=1}^{s}\frac{A_n^{\alpha-j}}{(1-e^{-it})^j} + \frac{u(\alpha-s-1,n,-t)}{(1-e^{-it})^s}\,e^{i(n+\frac{1}{2})t}\right] =$$

$$= \frac{1}{A_n^\alpha}\,\Im\left[-\frac{e^{-\frac{1}{2}it}}{2\sin\frac{1}{2}t}\sum_{j=1}^{s}\frac{A_n^{\alpha-j}}{(1-e^{-it})^j} + \frac{e^{i(n+\frac{1}{2})t}}{2\sin\frac{1}{2}t\cdot(1-e^{-it})^\alpha} - \right.$$

$$\left. - \frac{\sum_{\nu=n+1}^{\infty} A_\nu^{\alpha-s-1}\, e^{-i(\nu-n-\frac{1}{2})t}}{2\sin\frac{1}{2}t\cdot(1-e^{-it})^s}\right],$$

provided that the last series converges. So far the value of s has not been defined. Now we take s so large that the last series differentiated r times is still absolutely convergent. It is sufficient to suppose that $s > \alpha + r$. Since $A_n^\gamma = O(n^\gamma)$, and since each of the expressions

$$\left|\frac{d^h}{dt^h}\left(\frac{1}{2\sin\frac{1}{2}t\cdot(1-e^{-it})^\gamma}\right)\right|, \qquad \left|\frac{d^h}{dt^h}\left(\frac{e^{-\frac{1}{2}it}}{2\sin\frac{1}{2}t\cdot(1-e^{-it})^\gamma}\right)\right|$$

($\gamma \geqslant 0$) is less than $C_1/t^{\gamma+h+1}$, we obtain that $A_n^\alpha|\{K_n^\alpha(t)\}^{(r)}|$ is less than the sum of three expressions

$$C_2\sum_{j=1}^{s}\frac{n^{\alpha-j}}{t^{j+1+r}}, \qquad C_3\sum_{\mu=0}^{r}\frac{n^\mu}{t^{\alpha+1+r-\mu}}, \qquad C_4\sum_{\mu=0}^{r}\frac{n^{\alpha-s+\mu}}{t^{s+r-\mu+1}},$$

and the second part of the lemma follows at once. If $t \geqslant 1/n$, the second sum is $\leqslant C_5\, n^r/t^{\alpha+1}$, and the third is $< C_6\, n^{\alpha-s+r}/t^{s+1}$. Hence

$$\int_{1/n}^{\pi} t^r\left|\frac{d^r}{dt^r}K_n^\alpha(t)\right|dt \leqslant C_7\int_{1/n}^{\pi}\left[\frac{n^{r-\alpha}}{t^{\alpha-r+1}} + \frac{n^{-s+r}}{t^{s-r+1}} + \sum_{j=1}^{s}\frac{n^{-j}}{t^{j+1}}\right]dt < C_8.$$

On the other hand, from the formula

$$K_n^\alpha(t) = \{\tfrac{1}{2}A_n^\alpha + A_{n-1}^\alpha\cos t + \ldots\}/A_n^\alpha$$

we easily deduce that $|\{K_n^\alpha(t)\}^{(r)}|$ does not exceed the expression $n^r A_n^{\alpha+1}/A_n^\alpha < C_9\, n^{r+1}$. It follows that $\int_0^{1/n} t^r|\{K_n^\alpha(t)\}^{(r)}|\,dt < C_9$, and we obtain the first part of the lemma with $C = C_8 + C_9$.

10.42. *Let the series*

$$(1) \qquad \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

be summable (C, α), $\alpha = 0, 1, 2, \ldots$, *for* $x = x_0$, *to sum* s. *Let* r *be an integer* $> \alpha + 1$, *and suppose that the series* (1) *integrated term by term* r *times converges, in the neighbourhood of* x_0, *to a function* $F(x)$. *Then* $F_{(r)}(x_0)$ *exists and is equal to* s[1]).

To fix ideas we suppose that r is even; for r odd the proof would be similar. Increasing α, if necessary, we may suppose that either $r = \alpha + 2$, or $r = \alpha + 3$. We have [2])

$$(2) \qquad F(x) = \frac{a_0 x^r}{2r!} + (-1)^{r/2} \sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n^r}.$$

Without loss of generality we may assume that $x_0 = 0$, $s = 0$, $a_0 = 0$. We may also assume that (1) is a purely cosine series; for the sine component of (2) is an odd function of x, and so its r-th symmetric derivative at the point 0 is equal to 0. Let us put $\gamma(u) = (\cos u)/u^r$, $s_n = s_n^0 = a_1 + a_2 + \ldots + a_n, \ldots$, $s_n^k = s_1^{k-1} + \ldots + s_n^{k-1}, \ldots$ Since $s_n^\alpha = o(n^\alpha)$, and so also $s_n^{\alpha-1} = o(n^\alpha)$, $s_n^{\alpha-2} = o(n^\alpha), \ldots$, Abel's transformation applied $(\alpha + 1)$ times gives

$$F(t) = (-1)^{r/2} t^r \sum_{n=1}^{\infty} a_n \gamma(nt) = (-1)^{r/2} t^r \sum_{n=1}^{\infty} s_n^\alpha \Delta^{\alpha+1} \gamma(nt),$$

where the $(\alpha+1)$-st difference $\Delta^{\alpha+1}$ is defined by the following conditions: for any sequence $\{u_m\}$ we write $\Delta u_n = \Delta^1 u_n = u_n - u_{n+1}$, $\Delta^j u_n = \Delta(\Delta^{j-1} u_n)$. It is well known that, if $u(x)$ is a function differentiable j times, x_0 and $h > 0$ are fixed numbers, and $u_n = u(x_0 + nh)$, $n = 1, 2, \ldots$, then

$$(3) \qquad \Delta^j u_n = (-1)^j h^j u^{(j)}(x_0 + nh + \theta j h), \qquad 0 < \theta < 1 \text{ [3])}.$$

Let

$$P(x) = \sum_{\nu=0}^{\frac{1}{2}r-1} (-1)^\nu \frac{x^{2\nu}}{(2\nu)!}, \qquad \lambda(x) = \frac{\cos x - P(x)}{x^r}.$$

---

[1]) See Plessner, *Trigonometrische Reihen*, p. 1381. This result is a generalization of a theorem of Riemann (§ 11.21). The series (2) is certainly convergent if, for example, $|a_n| + |b_n| = o(n^\alpha)$.

[2]) Into the right-hand side of (2) we might introduce an arbitrary polynomial of order $(r-1)$; this would not affect the result.

[3]) The proof of (3) will be found in many treatises of Analysis. See e. g. de la Vallée-Poussin, *Cours d'Analyse*, 1, p. 72.

Then $\gamma(nt) = \lambda(nt) + P(nt)/(nt)^r$, and so

$$F(t) = \sum_{\nu=0}^{\frac{1}{2}r-1} \frac{A_\nu}{(2\nu)!} t^{2\nu} + t^r R(t),$$

where $A_\nu = (-1)^{\frac{1}{2}r+\nu} \Sigma s_n^\alpha \Delta^{\alpha+1} n^{2\nu-r}$, $R(t) = (-1)^{r/2} \Sigma s_n^\alpha \Delta^{\alpha+1} \lambda(nt)$. Since, in view of (3), $\Delta^{\alpha+1} n^{2\nu-r} = O(n^{2\nu-r-\alpha-1}) = O(n^{-\alpha-3})$, the series defining the numbers A_ν converge absolutely; it follows that the series defining $R(t)$ is also absolutely convergent. The theorem will have been established when we have shown that $R(t) = o(1)$ as $t \to 0$. Let $N = [1/t]$, $0 < t \leqslant 1$. Then

$$|R(t)| = \sum_{n=1}^{\infty} |s_n^\alpha \Delta^{\alpha+1} \lambda(nt)| = \sum_{n=1}^{N} + \sum_{n=N+1}^{\infty} = U + V.$$

The function $\lambda(u)$ is regular in the whole plane, and so, on account of (3), $|\Delta^{\alpha+1}\lambda(nt)| \leqslant Ct^{\alpha+1}$ for $n \leqslant N$, where $C, C_1, \ldots$ denote constants independent of n and t. It follows that U does not exceed $Ct^{\alpha+1}\sum_{n=1}^{N} |s_n^\alpha| = Ct^{\alpha+1} \cdot o(N^{\alpha+1}) = o(1)$ as $t \to 0$. On the other hand, an easy calculation shows that $|\gamma^{(\alpha+1)}(u)| \leqslant C_1 u^{-r}$, and so $|\lambda^{(\alpha+1)}(u)| \leqslant C_2 u^{-r}$, for $u \geqslant 1$. Using (3) again, we therefore obtain

$$V \leqslant C_2 t^{\alpha-r+1} \sum_{n=N+1}^{\infty} |s_n^\alpha| n^{-r} = C_2 t^{\alpha-r+1} \sum_{n=N+1}^{\infty} o(n^{\alpha-r}) = C_2 t^{\alpha-r+1} \cdot o(N^{r-\alpha-1}).$$

Hence $V = o(1)$, $U + V = o(1)$, and the theorem follows.

10.43. An immediate corollary of Theorems 10.41 and 10.42 is:

(i) *Suppose that the series* 10.42(1) *has coefficients* $O(n^k)$ *for some* k. *A necessary and sufficient condition that the series should be summable* C *for* $x = x_0$, *to sum* s, *is that there should exist an integer* $r > 0$ *such that, if* $F(x)$ *is the function obtained by integrating* 11.42(1) *term by term* r *times, then* $F_{(r)}(x_0)$ *exists and is equal to* s [1]).

When 10.42(1) is a Fourier series, the above result may be stated in a different form.

Given a function $\varphi(t)$, defined to the right of $t = 0$, we shall say that the number s is the (C, r) limit of $\varphi(t)$ as $t \to 0$, if

(1) $$\frac{r}{t^r}\int_0^t \varphi(u)(t-u)^{r-1}\,du \to s \quad \text{as} \quad t \to 0 \quad (r > 0).$$

[1]) Plessner, loc. cit.

A more detailed discussion of this notion will be found in § 12.3. The relation (1) will be written $(C, r)\,\varphi(t) \to s$. If $(C, \alpha)\,\varphi(t) \to s$ for some α, we shall write $(C)\,\varphi(t) \to s$.

(ii) *A necessary and sufficient condition that the Fourier series of a function $f(x)$ should be summable C, for $x = x_0$, to the sum $f(x_0)$, is $(C)\,\varphi_{x_0}(t) \to f(x_0)$, where $\varphi_{x_0}(t) = \frac{1}{2}[f(x_0+t) + f(x_0-t)]$*[1].

Let 10.42(1) be $\mathfrak{S}[f]$. Since $\mathfrak{S}[f]$ at the point $x = x_0$ is the same thing as $\mathfrak{S}[\varphi_{x_0}(t)]$ at the point $t = 0$, we may assume that $x_0 = 0$ and that $f(t)$ is an even function of t; we also assume that $f(0) = 0$. Fourier series may be integrated term by term, and so, if $F(x)$ is the result of integrating $\mathfrak{S}[f]$ r times, we have an equation

$$(2)\qquad F(t) + P(t) = \frac{1}{(r-1)!}\int_0^t \varphi(u)(t-u)^{r-1}\,du,$$

where $P(t)$ is a polynomial of order $\leq r-1$, and $\varphi(u) = \varphi_{x_0}(u) = f(u)$. From this we see that, if $(C, r)\,\varphi(t) \to 0$ as $t \to 0$, then $F_{(r)}(0)$ exists and is equal to 0. Conversely, if $F_{(r)}(0)$ exists and is equal to 0, then $F(t) = o(t^r) +$ a polynomial of order $r-2$; since the right-hand side of (2) is, in any case, $o(t^{r-1})$, it must be $o(t^r)$, i. e. $(C, r)\,\varphi_0(t) \to 0$. To complete the proof of (ii), we apply (i).

Proposition (ii) may be precised as follows.

(iii) *If $(C, \alpha)\,\varphi_{x_0}(t) \to f(x_0)$ as $t \to 0$, then $\mathfrak{S}[f]$ is summable (C, β), for $x = x_0$, to the value $f(x_0)$, where $\beta > \alpha \geq 0$.*

(iv) *If $\mathfrak{S}[f]$ is summable (C, β) to the sum $f(x_0)$, for $x = x_0$, then $(C, \alpha)\,\varphi_{x_0}(t) \to f(x_0)$ as $t \to 0$, where $\beta > -1$, $\alpha > \beta + 1$.*

For the proofs we refer the reader to Bosanquet [1], where also a further bibliography will be found. Here we intend to apply proposition (ii) to obtain an important result due to Hardy and Littlewood. For the proof we require the following theorem:

10.44. *If Σu_n is finite (C, α) and summable (C, β), $\beta > \alpha > -1$, then it is summable $(C, \alpha + \delta)$ for any $\delta > 0$*[2].

We may suppose that $\beta = \alpha + 1$, $0 < \delta < 1$, for the general result can be obtained by repeated application of this special case. Assuming, as we may, that the sum of Σu_n is 0, we have to prove that, with the notation of § 3.11, $s_n^{\alpha+\delta}/A_n^{\alpha+\delta} \to 0$. Now

---

[1] Hardy and Littlewood [7].

[2] Andersen [1].

$$s_n^{\alpha+\delta}=\sum_{k=0}^{n}A_{n-k}^{\delta-1}s_k^{\alpha}=\sum_{k=0}^{[n\vartheta]}+\sum_{k=[n\vartheta]+1}^{n}=P_n+Q_n \qquad (1/2<\vartheta<1).$$

Observing that $|s_k^{\alpha}|<C_1k^{\alpha}$, where $C_1, C_2, \dots$ denote constants, we have

$$|Q_n|\leqslant C_1 n^{\alpha}\sum_{k=[n\vartheta]+1}^{n}A_{n-k}^{\delta-1}=C_1 n^{\alpha}A_{n-[n\vartheta]-1}^{\delta}\simeq C_2 n^{\alpha+\delta}(1-\vartheta)^{\delta}.$$

(Since $\vartheta>\frac{1}{2}$, the first inequality is true for $\alpha<0$ also). Hence, if ϑ is sufficiently near to 1, we have $|Q_n|/A_n^{\alpha+\delta}<\frac{1}{2}\varepsilon$, where ε is arbitrarily given and $n>n_0$. Having fixed ϑ, we shall prove that $P_n=o(n^{\alpha+\delta})$; for, making Abel's transformation,

$$|P_n|\leqslant|\sum_{k=0}^{[n\vartheta]}A_{n-k}^{\delta-2}s_k^{\alpha+1}|+o(n^{\alpha+\delta})<C_3[n(1-\vartheta)]^{\delta-2}\sum_{k=0}^{[n\vartheta]}o(k^{\alpha+1})+o(n^{\alpha+\delta})=$$

$$=C_3[n(1-\vartheta)]^{\delta-2}o(n^{\alpha+2})=o(n^{\alpha+\delta})<\tfrac{1}{2}\varepsilon A_n^{\alpha+\delta}$$

for $n>n_1$. Hence $|s_n^{\alpha+\delta}/A_n^{\alpha+\delta}|<\varepsilon$ for $n>\mathrm{Max}(n_0,n_1)$, and the theorem is established.

10.45 [1]. (i) *If f is non-negative and if $\mathfrak{S}[f]$ is summable C at a point x, then $\mathfrak{S}[f]$ is summable (C,ε) at that point, for every positive ε.*

(ii) *If $f\geqslant 0$, a necessary and sufficient condition that $\mathfrak{S}[f]$ should be summable C at a point x, to $f(x)$, is that $(C,1)\,\varphi_x(t)\to f(x)$.*

Under the hypothesis of (i), we have 10.43(1), with $\varphi(u)=\varphi_x(u)$, for some $r>0$. Since $\varphi_x(u)\geqslant 0$, the left-hand side of 10.43(1) is not less than

$$\frac{r}{t^r}\int_0^{t/2}\varphi_x(u)(t-u)^{r-1}du\geqslant\frac{r\,2^{1-r}}{t}\int_0^{t/2}\varphi_x(u)\,du,\ \text{i. e.}\ \frac{1}{t}\int_0^t\varphi_x(u)\,du=O(1).$$

Let $\xi_x(t)=\frac{1}{t}\int_0^t|f(x+t)+f(x-t)-2f(x)|\,dt$. In § 3.3 we proved that, at any point x where $\xi_x(t)=o(1)$, $\mathfrak{S}[f]$ is summable (C,α), $\alpha>0$, to the sum $f(x)$. Exactly the same argument shows that, if $\xi_x(t)=O(1)$, then $\mathfrak{S}[f]$ is finite (C,α) at the point x (it must be remembered that $\varphi_x(t)$ has a slightly different meaning in § 3.3, viz. $f(x+t)+f(x-t)-2f(x)$). Since the conditions $\varphi_x(t)\geqslant 0$ and $\frac{1}{t}\int_0^t\varphi_x(u)\,du=O(1)$ imply $\xi_x(t)=O(1)$, $\mathfrak{S}[f]$ is, in our case, finite

[1] Hardy and Littlewood [5].

(C, α) and so, in view of Theorem 10.44, summable $(C, \alpha+\delta)$ for every $\alpha > 0$ and $\delta > 0$. Putting $\alpha + \delta = \varepsilon$, we obtain (i).

We write $\varphi_x(t) = \varphi(t) = \Phi_0(t)$, and denote by $\Phi_k(t)$, $k=1,2,\dots$, the integral of $\Phi_{k-1}(u)$ over $0 \leqslant u \leqslant t$. The relation 10.43(1), with $s = f(x)$, may be written $\Phi_r(t) \simeq f(x)\, t^r/r!$, and to prove (ii) we have to show that $\Phi_1(t) \simeq f(x)\, t$. Since $\Phi_k(t)$, $k = 1, 2, \dots$, is a non-decreasing function of t, proposition (ii) follows by repeated application of the following lemma:

Let $s(t)$, $t \geqslant 0$, be an everywhere differentiable function of t. If $s'(t)$ is non-decreasing and if $s(t) \simeq st^\alpha$ as $t \to 0$, then $s'(t) \simeq s\alpha t^{\alpha-1}$.

Let $0 < \theta < 1$ be a fixed number; by the mean-value theorem,

$$(1) \qquad (1-\theta)\, t\, s'(\theta t) < s(t) - s(\theta t) < (1-\theta)\, t\, s'(t).$$

Since $s(t) - s(\theta t) \simeq s \cdot (1 - \theta^\alpha)\, t^\alpha$, we obtain from (1)

$$\underline{\lim}_{t \to 0} \frac{s'(t)}{t^{\alpha-1}} \geqslant s \frac{1-\theta^\alpha}{1-\theta},$$

$$\overline{\lim}_{t\to 0} \frac{s'(\theta t)}{(\theta t)^{\alpha-1}} \leqslant s \frac{1-\theta^\alpha}{(1-\theta)\,\theta^\alpha}, \text{ i. e. } \overline{\lim}_{t\to 0} \frac{s'(t)}{t^{\alpha-1}} \leqslant s \frac{1-\theta^\alpha}{(1-\theta)\,\theta^{\alpha-1}}.$$

Since θ may be taken as near to 1 as we please, we obtain $\underline{\lim}\, s'(t)/t^{\alpha-1} \geqslant s\alpha$, $\overline{\lim}\, s'(t)/t^{\alpha-1} \leqslant s\alpha$, i. e. $s'(t) \simeq s\alpha t^{\alpha-1}$.

It is plain that (i) and (ii) hold when f is bounded below, and so, in particular, when f is bounded.

10.46. Miscellaneous theorems and examples.

1. If $f \in L^r$, $r > 1$, and if $s_n(x)$ are the partial sums of $\mathfrak{S}[f]$, then

$$\frac{1}{n+1} \sum_{\nu=0}^{n} e^{\lambda |f(x) - s_\nu(x)|} \to 1$$

for every $\lambda > 0$ and almost every x. In particular, $s_n(x) = o(\log n)$ for almost every x. Carleman [2].

[Use the equation $e^{\lambda u} = 1 + \lambda u + \dots$ and argue as in § 10.1].

2. If $f \in L^2$ and $s_n(x)$ are the partial sums of $\mathfrak{S}[f]$, then, for almost every x, the sequence $1, 2, 3, \dots$ can be broken up into two complementary sequences $\{m_k\}$ and $\{n_k\}$ (depending in general on x) such that $s_{m_k}(x) \to f(x)$, $\Sigma\, 1/n_k < \infty$.

[Use the lemma of § 10.11].

3. A series Σu_n is said to be *absolutely summable A*, if the function $g(r) = \Sigma u_n r^n$ is of bounded variation over $0 \leqslant r < 1$. Show that, if $\Sigma |u_n| < \infty$ then Σu_n is absolutely summable A.

4. $\mathfrak{S}[f]$ is absolutely summable A for $x = x_0$ provided that either (i) f satisfies Dini's test (§ 2.4) at the point x_0, or (ii) $f(x)$ is of bounded variation in the neighbourhood of x_0. See Whittaker [1], Prasad [2].

5. Let $s_n(x)$ and $\bar{s}_n(x)$ denote the partial sums of $\mathfrak{S}[f]$ and $\bar{\mathfrak{S}}[f]$ respectively, and suppose that there is a function $g(x) \geqslant 0$, $g \epsilon L$, such that $s_n(x) \geqslant -g(x)$, $n=0,1,2,\ldots$ Then (i) there is a function $h(x)$ belonging to $L^{1-\varepsilon}$ for every $\varepsilon > 0$ and such that $s_n(x) \leqslant h(x)$, $|\bar{s}_n(x)| \leqslant h(x)$. Moreover, (ii) if $f \epsilon L^r$, $g \epsilon L^r$, $r > 1$, then $h \epsilon L^r$, (iii) if $f \log^+ |f| \epsilon L$, $g \log^+ |g| \epsilon L$, then $h \epsilon L$.

For this and the following theorem see Paley and Zygmund [2].

6. (i) If $|f| \leqslant 1$, and $s_n(x) > -A$, $0 \leqslant x \leqslant 2\pi$, $n=0,1,\ldots$, where A is a constant, then there is a constant $B = B(A)$ such that $s_n(x) \leqslant B$. (ii) If $f(x)$ is continuous and, for any $\varepsilon > 0$, we have $s_n(x) > f(x) - \varepsilon$, $n \geqslant n(\varepsilon)$, $0 \leqslant x \leqslant 2\pi$, then $\mathfrak{S}[f]$ converges uniformly to $f(x)$.

7. Let $\{a_n\}$ be a positive decreasing sequence such that $\{na_n\}$ is monotonic and $\Sigma a_n/n < \infty$; let $s_n(x)$ and $t_n(x)$ denote the partial sums of the series $\Sigma a_n \cos nx$ and $\Sigma a_n \sin nx$ respectively; then the functions $s(x) = \operatorname{Sup}_n |s_n(x)|$ and $t(x) = \operatorname{Sup}_n |t_n(x)|$ are both integrable.

8. If a_n and b_n, $n = 1, 2, \ldots$, are the Fourier coefficients of an integrable function, the partial sums of the series

$$\sum_{n=2}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{(\log n)^{1+\delta}}, \qquad \sum_{n=2}^{\infty} \frac{a_n \sin nx - b_n \cos nx}{(\log n)^{1+\delta}} \qquad (\delta > 0)$$

can be majorized by integrable functions. For $\delta = 0$ this is no longer true.

9. If $a_k \geqslant 0$, $k = 1, 2, \ldots$, and if the series $\Sigma a_k \sin kx$ is the Fourier series of a bounded function $f(x)$, the partial sums of the series are uniformly bounded; if f is continuous, the series converges uniformly. Paley [7].

[Let $\sigma_n(x)$ be the first arithmetic means of the series considered. To prove the first part of the theorem, observe that, if $|f(x)| \leqslant M$, then $|\sigma_{2n}(x)| \leqslant M$, $|\sigma'_{2n}(x)| \leqslant 4n\,M$ (§ 7.31), and so, taking $x = 0$,

$$\sum_{k=1}^{2n} \left(1 - \frac{k}{2n+1}\right) k a_k \leqslant 4Mn.$$

Taking the first n terms on the left, we obtain $a_1 + 2a_2 + \ldots + na_n \leqslant 8Mn$, and it is sufficient to apply 3.13(1)].

10. Theorem 10.42 holds for α fractional and > -1.

[For $-1 < \alpha < 0$, $r = 1$, the theorem was established by Hardy and Littlewood [1]. The general result can be obtained by combining the Hardy-Littlewood argument with that of § 10.42].

11. The results concerning summability C holds, *mutatis mutandis*, for Fourier-Stieltjes series; in particular, if $F(x)$ is non-decreasing, summability C involves summability (C, ε) for any $\varepsilon > 0$; a necessary and sufficient condition that $\mathfrak{S}[dF]$ should be summable C for $x = x_0$, is that $F_{(1)}(x_0)$ should exist.

12. Power series on the circle of convergence may be considered as trigonometrical series, so that Theorem 10.43(i) remains true for power series

$$(1) \qquad \sum_{n=0}^{\infty} \alpha_n e^{inx}.$$

It may however then be stated in a slightly different form, viz. it holds if by $F_{(r)}(x_0)$ we mean the r-th *unsymmetric* generalized derivative defined at the beginning of § 10.41. Plessner. *Trigonometrische Reihen*, p. 1382; see also Hardy and Littlewood [7].

[Theorem 10.42 holds if $\alpha > -1$, $r > \alpha + 1$, and $F_{(r)}(x)$ is the r-th unsymmetric generalized derivative, provided that 10.42(1) is of the form (1)].

13. If 10.42(1) is the Fourier series of a bounded function $f(x)$, the conjugate series is summable C if and only if it is summable (C, ε) for every $\varepsilon > 0$. A necessary and sufficient condition that $\bar{\mathfrak{S}}[f]$ should be summable C for $x = x_0$, is the existence of the integral

$$-\frac{1}{\pi}\int_0^{\pi} \frac{f(x_0+t) - f(x_0 - t)}{2\,\mathrm{tg}\,\frac{1}{2}t}\, dt,$$

which represents, then, the sum of $\bar{\mathfrak{S}}[f]$ for $x = x_0$. Prasad [3], Hardy and Littlewood [19].

[To prove the first part of the theorem, we show that the difference 3.32(1) is bounded for every $r > 0$ (that it is bounded for $0 < r \leqslant 1$, was implicitely proved in 3.32). For then $\bar{\sigma}_n^r(x_0) - \bar{\sigma}_n^s(x_0) = O(1)$ for every $r > 0$ and $s > 0$, and it is sufficient to apply Theorem 10.44. For the second part of the theorem we refer the reader to the papers quoted [1])].

---

[1]) A theory of summability C of the series conjugate to general trigonometrical series will be found in Plessner, loc. cit.

CHAPTER XI.

Riemann's theory of trigonometrical series.

11.1. In the previous chapters we have, almost exclusively, considered the behaviour of Fourier series. Now we shall prove a number of theorems concerning the properties of trigonometrical series

$$(1) \qquad \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

with coefficients tending to 0, but otherwise quite arbitrary. The fundamental results in this field are due to Riemann, and these results, with their subsequent extensions, constitute what is now called the Riemann theory of trigonometrical series. The chief results of the Riemann theory concern the problems of *uniqueness* and of *localization* for trigonometrical series.

In what follows we shall suppose, unless otherwise stated, that the coefficients of the trigonometrical series considered tend to 0.

11.11. The Cantor-Lebesgue theorem. In the sequel we shall frequently use the following notation:

$$\tfrac{1}{2} a_0 = A_0(x), \quad a_n \cos nx + b_n \sin nx = A_n(x),$$

$$b_n \cos nx - a_n \sin nx = B_n(x), \quad n > 0,$$

$$A_n(x) = \rho_n \cos(nx + \alpha_n), \text{ where } \rho_n^2 = a_n^2 + b_n^2, \quad \rho_n \geqslant 0.$$

The following theorem is called the Cantor-Lebesgue theorem:

(i) *If $A_n(x)$ tends to 0, as $n \to \infty$, for every x belonging to a set E of positive measure, then $a_n \to 0$, $b_n \to 0$.*

For, if ρ_n does not tend to 0, there exists a sequence $n_1 < n_2 < \dots$ of indices, and an $\varepsilon > 0$ such that $\rho_{n_k} > \varepsilon$, $k = 1, 2, \dots$ From this, and the relation $\rho_n \cos(nx + \alpha_n) \to 0$, we obtain that $\cos(n_k x + \alpha_{n_k}) \to 0$ and, à fortiori, $\cos^2(n_k x + \alpha_{n_k}) \to 0$ for $x \in E$. The terms of the last sequence do not exceed 1, and so, by Lebesgue's theorem on the integration of bounded sequences, the expression

$$(1) \quad \int_E \cos^2(n_k x + \alpha_{n_k})\, dx = \tfrac{1}{2} \int_E dx + \tfrac{1}{2} \int_E \cos 2\,(n_k x + \alpha_{n_k})\, dx$$

tends to 0. Since the numbers $\dfrac{1}{\pi} \displaystyle\int_E \cos 2n_k x\, dx$, $\dfrac{1}{\pi} \displaystyle\int_E \sin 2n_k x\, dx$ are Fourier coefficients of the characteristic function of E, they tend to 0, and the right-hand side of the equation (1) tends to $\frac{1}{2}|E| > 0$. This contradiction proves the theorem. As corollaries we obtain the following propositions, the second of which contains the first as a special case.

(ii) *If the series* 11.1(1) *converges in a set E of positive measure, then $a_n \to 0$, $b_n \to 0$.*

(iii) *If the series* 11.1(1) *is summable* (C, k), $k > -1$, *in a set E of positive measure, then $a_n = o(n^k)$, $b_n = o(n^k)$.*

To prove (iii), we observe that $a_n n^{-k} \cos nx + b_n n^{-k} \sin nx \to 0$ for $x \in E$ (§ 3.13) and apply (i). From (iii) and Theorem 2.221 we infer that, in the general case, the method (C, k), $k < 1$, is too weak to sum Fourier-Denjoy series.

11.12. A generalization of the previous theorem. Given any sequence of real numbers $\alpha_1, \alpha_2, \dots$, and a number $-1 < \beta < 1$, we shall denote by E_n the set of points in the interval $(0, 2\pi)$ for which $\cos(nx + \alpha_n) \geqslant \beta$. We have $|E_n| = 2\pi\theta$, where the positive number θ is equal to $(\arccos \beta)/\pi$, and so $|E_n|$ depends on β only.

For any infinite sequence $n_1 < n_2 < \dots$, and fixed β, the product $E = E_{n_1} E_{n_2} \dots$ is of measure 0. Clearly we may omit as many factors in the product as we please, since this only extends E. In the first place, we observe that, if S is any finite system of intervals, then $|SE_n| \to \theta |S|$ as $n \to \infty$. Now let $\theta < \theta_1 < 1$, $m_1 = n_1$, and suppose that we have already defined $m_1, m_2, \dots, m_{k-1}$. If $S_{k-1} = E_{m_1} E_{m_2} \dots E_{m_{k-1}}$, we can find a number $m_k > m_{k-1}$ belonging to $\{n_i\}$ and such that $|S_{k-1} E_{m_k}| \leqslant \theta_1 |S_{k-1}|$. Hence, putting

$S_k = E_{m_1} E_{m_2} \dots E_{m_k}$, we have $|S_k| \leqslant 2\pi\theta_1^k$. Therefore $|E_{m_1} E_{m_2} \dots| = 0$ and, à fortiori, $|E| = 0$.

Sets such as the set E which we have just considered, will be called H-sets[1]). Every H-set is defined by the sequences $n_1, n_2, \dots, \alpha_{n_1}, \alpha_{n_2}, \dots$ (the second of which we may denote by $\alpha_1, \alpha_2, \dots$ simply) and the number β. If $n_k = 3^k$, $\alpha_k = 0$, $k = 1, 2, \dots$, and $\beta = -\frac{1}{2}$, we obtain Cantor's ternary set constructed on $(0, 2\pi)$.

We shall say that a set is a H_σ-set if it is a sum of a finite or enumerable sequence of H-sets. Since every H-set is closed and of measure 0, it is non-dense. Therefore sets of type H_σ are of the first category and of measure 0.

We shall require the following lemma.

If $\{\alpha_k\}$ is an arbitrary sequence of real numbers and $n_1 < n_2 < \dots$ an arbitrary sequence of integers, then, except perhaps for x belonging to a set E of type H_σ, we have $\overline{\lim}\,|\cos(n_k x + \alpha_k)| = 1$.

If $0 < \gamma < 1$ and if $|\cos(n_k x + \alpha_k)| \leqslant \gamma$, then, à fortiori $\cos(n_k x + \alpha_k) \geqslant -\gamma$. Let $G_i^{(\gamma)}$ denote the set of x such that $|\cos(n_k x + \alpha_k)| \leqslant \gamma$ for $k \geqslant i$. From what we have just said it follows that $G_i^{(\gamma)} \subset F_i^{(\gamma)}$, where $F_i^{(\gamma)}$ is an H-set. Therefore $G^{(\gamma)} = G_1^{(\gamma)} + G_2^{(\gamma)} + \dots$ is contained in an H_σ-set, and the same is true for the set $E = G^{(1/2)} + G^{(2/3)} + G^{(3/4)} + \dots$, outside which we have $\overline{\lim}\,|\cos(n_k x + \alpha_k)| = 1$.

Now we are in a position to prove the following theorem due to Steinhaus.

Except perhaps in a set E of measure 0 and of type H_σ,

$$\overline{\lim_{n\to\infty}}\,|a_n \cos nx + b_n \sin nx| = \overline{\lim_{n\to\infty}}\,\sqrt{a_n^2 + b_n^2}\ ^{2)}.$$

Let $A_n(x) = \rho_n \cos(nx + \alpha_n)$, and let $\{n_k\}$ be a sequence such that $\lim \rho_{n_k} = \overline{\lim}\,\rho_n$. If E is the set E of the lemma, then outside E we have

$$\overline{\lim}\,|A_n(x)| \geqslant \overline{\lim}\,|A_{n_k}(x)| = \lim \rho_{n_k} = \overline{\lim}\,\rho_n,$$

i. e. $\overline{\lim}\,|A_n(x)| \geqslant \overline{\lim}\,\rho_n$. Since the inverse inequality is satisfied for every x, the theorem follows.

---

[1]) These sets were introduced by Rajchman [1].

[2]) Steinhaus [8] proved that $|E| = 0$; that E is of type H_σ was shown by Rajchman [1].

It is plain that the Cantor-Lebesgue theorem is a consequence of Steinhaus's. Since H_σ-sets are of the first category, we obtain, in particular, that, *if $A_n(x)$ tends to 0 in a set of the second category, then $a_n \to 0$, $b_n \to 0$* [1]).

11.2. Riemann's theorems on the formal integration of trigonometrical series [2]). Given the series 11.1(1) with $a_n, b_n \to 0$, consider the function

$$(1) \qquad F(x) = {}^1\!/_4\, a_0 x^2 - \sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n^2}$$

The series on the right, which is obtained by integrating 11.1(1) formally twice, converges absolutely and uniformly, and so $F(x)$ is continuous. It will be readily seen that

$$(2) \qquad \frac{F(x+2h)+F(x-2h)-2F(x)}{4h^2} = A_0 + \sum_{n=1}^{\infty} A_n(x)\left(\frac{\sin nh}{nh}\right)^2.$$

The numerator of the ratio on the left will be denoted by $\Delta^2 F(x, 2h)$. The upper and lower limits of indetermination of $\Delta^2 F(x, h)/h^2$, as $h \to 0$, will be denoted by $\overline{D}^2 F(x)$ and $\underline{D}^2 F(x)$ respectively. The common value of $\overline{D}^2 F(x)$ and $\underline{D}^2 F(x)$, if it exists, will be denoted by $D^2 F(x)$ and called the *generalized second derivative* of F at the point x. If $D^2 F(x_0)$ exists and is finite, we shall say that the series 11.1(1) is, at the point x_0, summable by Riemann's method of summation, or summable R, to the value $D^2 F(x)$.

(i) *If 11.1(1), where $a_n, b_n \to 0$, converges at a point x to sum s, it is also summable R to the same sum.*

It is sufficient to show that $\Delta^2 F(x, 2h_i)/4h_i^2$ tends to s for every sequence $\{h_i\}$ of positive numbers tending to 0. Let us put $A_0 + A_1 + \ldots + A_n = s_n$, $(\sin^2 h)/h^2 = u(h)$. Applying Abel's transformation, we see that the right-hand side of (2), for $h = h_i$, is equal to

$$(3) \qquad \sum_{n=0}^{\infty} s_n \cdot \{u(nh_i) - u((n+1)h_i)\}.$$

---

1) Young [14].

2) Riemann [1]. Proposition (i) of this section is a special case of Theorem 10.42, but we prefer not to use that result.

Here we have a linear transformation of the sequence $s_n \to s$, and, to prove that (3) tends to s, it is sufficient to show that the Toeplitz conditions of § 3.1 are satisfied. Conditions (i) and (ii) are obviously satisfied. To verify (iii) we observe that

$$(4) \qquad \sum_{n=0}^{\infty} |u(nh_i) - u((n+1)h_i)| \leqslant \sum_{n=0}^{\infty} \int_{nh_i}^{(n+1)h_i} |u'(t)|\, dt = \int_0^{\infty} |u'(t)|\, dt,$$

and that the last integral is finite.

Theorem (i) may be generalized as follows.

(i') *If the series* 11.1(1) *has partial sums* $s_n(x)$ *bounded at* x, *and if* $\underline{s}(x) = \underline{\lim}\, s_n(x)$, $\bar{s}(x) = \overline{\lim}\, s_n(x)$, *then the numbers* $\overline{D}^2 F(x)$ *and* $\underline{D}^2 F(x)$ *are both contained in the interval* $(s - k\delta,\ s + k\delta)$, *where* $2s = \underline{s}(x) + \bar{s}(x)$, $2\delta = \bar{s}(x) - \underline{s}(x)$, *and* k *is an absolute constant.*

This follows from § 3.101, if for k we take the upper bound, for all $\{h_i\}$, of the sums on the left of (4).

(ii) *If* a_n *and* b_n *tend to* 0, *then*

$$(5) \qquad \frac{F(x+2h) + F(x-2h) - 2F(x)}{4h} = A_0 h + \sum_{n=1}^{\infty} A_n \frac{\sin^2 nh}{n^2 h} \to 0$$

as $h \to 0$.

It is again sufficient to prove (5) for any sequence $\{h_i\}$ of positive numbers tending to 0. The series in (5) is a linear transformation of the sequence $A_n \to 0$, and so it is sufficient to verify Toeplitz's conditions (i) and (iii) (condition (ii) need not be tested). The first of them is obviously satisfied. To prove (iii) we observe that

$$(6) \quad h_i + \sum_{n=1}^{\infty} \frac{\sin^2 nh_i}{n^2 h_i} \leqslant h_i + \sum_{n=1}^{N} \frac{n^2 h_i^2}{n^2 h_i} + \sum_{n=N+1}^{\infty} \frac{1}{n^2 h_i} < (N+1)\, h_i + 1/Nh_i.$$

If we put $N = [1/h_i] + 1$, then $1/h_i < N \leqslant 1/h_i + 1$ and the right-hand side of (6) is less than 4 for $|h_i| \leqslant 1$. This completes the proof.

It is plain that (5) is satisfied uniformly in x.

The relation (5) is satisfied at every point x, irrespectively of the convergence or divergence of the series 11.1(1). If $G(x)$ is the sum of an arbitrary trigonometrical series with coefficients $o(n^{-2})$, then $\Delta^2 G(x, 2h)/4h \to 0$ for every x, and $h \to 0$; for G may be considered as the function F corresponding to a trigonometrical series with coefficients tending to 0.

If for a function $F(x)$ we have $\Delta^2 F(x_0, h)/h \to 0$, then F will be said to be *smooth* at the point x_0. For, writing $\Delta^2 F(x_0, h)/h$ in the form $\{F(x_0+h) - F(x_0)\}/h - \{F(x_0) - F(x_0-h)\}/h$, we see that F cannot have an angular point at x_0: if the right-hand and the left-hand derivatives at x_0 exist, they must be equal.

11.21. Fatou's theorems. Instead of the function $F(x)$ defined by 11.2(1), we may consider the function

$$(1) \qquad L(x) = \tfrac{1}{2} a_0 x + \sum_{n=1}^{\infty} \frac{a_n \sin nx - b_n \cos nx}{n}$$

obtained from 11.1(1) by a single integration. Then

$$\frac{L(x+h) - L(x-h)}{2h} = A_0 + \sum_{n=1}^{\infty} A_n(x) \left(\frac{\sin nh}{nh}\right).$$

The trouble is that, in the general case, the series in (1) need not converge everywhere, even if 11.1(1) converges for every x (a simple example is provided by the series $\Sigma(\sin nx)/\log n$), and this makes the function $L(x)$ much less convenient in applications.

If $L(x)$ exists in a neighbourhood of a point x_0 and if the ratio $\{L(x_0+h) - L(x_0-h)\}/2h$ tends to a limit s as $h \to 0$, we shall say that the series 11.1(1) is summable by Lebesgue's method of summation, or summable L, to the value s, at the point x_0.

(i) *If a_n and b_n are $o(1/n)$, a necessary and sufficient condition that the series* 11.1(1) *should converge, at a point x, to sum s, is that it should be summable L to s* [1]).

In view of the conditions imposed upon a_n and b_n, $F(x)$ exists for every x. Let $s_n(x) = A_0 + A_1 + \dots + A_n$, $N = [1/h]$; then

$$(2) \qquad \frac{L(x+h) - L(x-h)}{2h} - s_N = \sum_{n=1}^{N} A_n \left(\frac{\sin nh}{nh} - 1\right) + \sum_{n=N+1}^{\infty} A_n \frac{\sin nh}{nh} = P + Q.$$

The terms of Q are $o(n^{-2} h^{-1})$, and so $Q = o(N^{-1} h^{-1}) = o(1)$. Since

---

[1]) Fatou [1]. In this proposition, as well as in (ii) below, the number s may be infinite.

$(\sin u)/u - 1 = O(u^2) = O(u)$ for $|u| \leqslant 1$, the terms of P are $h \cdot o(1)$, and $P = o(Nh) = o(1)$. Therefore $P + Q = o(1)$, and, in fact, uniformly in x, and the theorem follows.

By the Riesz-Fischer theorem, trigonometrical series with coefficients $o(1/n)$ are Fourier series.

(ii) *If a_n and b_n are $o(1/n)$, and if* 11.1(1) *is the Fourier series of a function f such that $f(x) \to s$ as $x \to x_0 + 0$, then the series converges at the point x_0 to the value s.*

(iii) *If a_n and b_n are $o(1/n)$ and if* 11.1(1) *is $\mathfrak{S}[f]$, where f is continuous in an interval $a \leqslant x \leqslant b$, then the series converges uniformly in that interval.*

To prove (ii) we observe that, at the point x_0, the function $L(x)$ has a right-hand derivative equal to s. Since $L(x)$ is a smooth function (§ 11.2), the left-hand derivative at x_0 exists and is also equal to s. Hence $\{L(x_0 + h) - L(x_0 - h)\}/2h \to s$, and so, by (i), $s_n(x_0) \to s$.

To prove (iii) we notice that, if $h \to +0$, then $\{L(x+h) - L(x)\}/h$ tends to $f(x)$, uniformly in the interval $(I)\ a \leqslant x \leqslant a + \frac{1}{2}(b-a)$. Since $\Delta^2 L(x, h)/h \to 0$ uniformly in x (§ 11.2), we obtain that $\{L(x) - L(x-h)\}/h \to f(x)$, and so also $\{L(x+h) - L(x-h)\}/2h \to f(x)$, uniformly in I. Similarly we prove the last relation in the remaining part of (a, b), and it is sufficient to observe that the left-hand side of (2) tends to 0 uniformly in x.

11.3. Uniqueness of trigonometrical series. In previous chapters we have learnt to associate with every integrable and periodic function $f(x)$ a special trigonometrical series — the Fourier series of $f(x)$ — which, as we have shown, represents $f(x)$ in various ways. It is natural to inquire whether functions can be represented by trigonometrical series other than Fourier series. This problem has many aspects, according to the sense which we assign to the word 'represent'. The problem of the convergence, or summability, in mean was discussed in Chapter IV. In this paragraph we shall consider the representation of functions by means of trigonometrical series which are everywhere convergent. The following results are fundamental for the theory of trigonometrical series.

(i) *If a trigonometrical series converges everywhere to 0, the series vanishes identically, i. e. all the coefficients are equal to 0.*

(ii) *If two trigonometrical series converge to the same sum in the interval* $(0, 2\pi)$, *the series are identical, i. e. corresponding coefficients in the two series are equal.*

(iii) *If a trigonometrical series converges in the interval* $(0, 2\pi)$ *to an integrable function* $f(x)$, *the series is* $\mathfrak{S}[f]$.

Of these theorems, (ii) follows from (i), and the latter is, in turn, a consequence of (iii). Theorem (i) is due to Cantor; (iii) was established, in the case of f bounded and integrable in the Riemann sense, by Du Bois-Reymond, and in the general case by de la Vallée Poussin [1]).

The most important step in the proof of (iii) will have been achieved when we have shown that the function $F(x)$ defined by 11.2(1) satisfies an equation

$$(1)\qquad F(x) = \int_a^x dy \int_a^y f(t)\,dt + Ax + B, \qquad (A, B \text{ constants})$$

i. e. that the formal integration of the series 11.1(1) corresponds to the integration of $f(x)$. For let $F_1(x) = F(x) - {}^1/_4\, a_0 x^2$; it is clear that the series 11.2(1) without the quadratic term is $\mathfrak{S}[F_1]$. The function $F_1(x)$ is a second integral and, as may be seen from 11.2(1), a periodic function. Let us put $2c_n = a_n - ib_n$ and write $\mathfrak{S}[F_1]$ in the complex form. Integrating by parts twice and observing that $F_1(x)$ and $F_1'(x)$ are periodic, we have, for $n \neq 0$,

$$-\frac{c_n}{n^2} = \frac{1}{2\pi}\int_0^{2\pi} F_1 e^{-inx}\,dx = -\frac{1}{2\pi n^2}\int_0^{2\pi} F_1'' e^{-inx}\,dx =$$

$$= -\frac{1}{2\pi n^2}\int_0^{2\pi} [f - \tfrac{1}{2} a_0]\, e^{-inx}\,dx = -\frac{1}{2\pi n^2}\int_0^{2\pi} f e^{-inx}\,dx,$$

i. e.

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f e^{-inx}\,dx.$$

---

[1]) Cantor [1], Du Bois-Reymond [3], de la Vallée-Poussin [3]; Denjoy [4] showed that, with a suitable definition of an integral, more general that that of Lebesgue, every trigonometrical series convergent to a finite sum is a Fourier series.

To find the same formula for $c_0 = \frac{1}{2} a_0$, it is sufficient to observe that the function $F_1'(x) = F'(x) - \frac{1}{2} a_0 x$ is periodic, and so the integral of $F_1''(x) = F''(x) - \frac{1}{2} a_0 = f(x) - \frac{1}{2} a_0$ over the interval $(0, 2\pi)$ is equal to 0.

11.31. We shall now prove a number of lemmas, which give a little more than we actually require.

(i) *If a continuous function $F(x)$, $a < x < b$, satisfies the inequality $\overline{D}^2 F(x) \geqslant 0$, except perhaps at an enumerable set E, where however F is smooth, then F is convex.*

It is sufficient to consider the case $\overline{D}^2 F > 0$ for, if we put $F_n(x) = F(x) + x^2/n$, then $\overline{D}^2 F_n(x) > 0$, $F_n(x) \to F(x)$, and the limit of a sequence of convex functions is convex. If $F(x)$ were not convex, there would exist two points α and β, and a linear function $l(x) = mx + n$, such that $\rho(x) = F(x) - l(x)$ would vanish for $x = \alpha$, $x = \beta$, and would assume positive values somewhere in (α, β). It is important to observe that, if we replace m by m_1, where $m_1 > m$ and $m_1 - m$ is sufficiently small, we shall still have the same situation. Let x_0 be a point in (α, β) where $\rho(x)$ attains its maximum; hence $\Delta^2 \rho(x_0, h) \leqslant 0$ for h positive and sufficiently small. It follows that $\overline{D}^2 \rho(x_0) = \overline{D}^2 F(x_0) \leqslant 0$, which contradicts our hypothesis, and so proves the lemma, unless $x_0 \in E$.

Suppose now that x_0 belongs to E, and divide the inequality $\Delta^2 \rho(x_0, h) = \rho(x_0 + h) - \rho(x_0) + \rho(x_0 - h) - \rho(x_0) \leqslant 0$ by $h \to +0$. The function $\rho(x)$ is smooth at x_0, and so, taking into account that $\rho(x_0 + h) - \rho(x_0) \leqslant 0$, $\rho(x_0 - h) - \rho(x_0) \leqslant 0$ for h small enough, we obtain that the right-hand and the left-hand derivatives of $\rho(x)$ at x_0 exist and are equal to 0, i. e. $\rho'(x_0) = F'(x_0) - m = 0$; in particular $F'(x_0)$ exists. Therefore if, instead of m, we take a number $m_1 > m$ sufficiently near to m, and such that $m_1 \neq F'(\xi)$ for every $\xi \in E$, the point x_0 does not belong to E, and in this case the lemma has already been established.

(ii) *If a function $F(x)$, $a < x < b$, has a continuous derivative $F'(x)$ and if, at a point x_0, all the derivates of $F'(x)$ are contained between m and M, then $m \leqslant \underline{D}^2 F(x_0) \leqslant \overline{D}^2 F(x_0) \leqslant M$.*

By the mean-value theorem, the ratio $\Delta^2 F(x_0, h)/h^2$ is equal to $[F'(x_0 + h_1) - F'(x_0 - h_1)]/2h_1$, $0 < h_1 < h$; and since the last ratio is the arithmetic mean of the expressions $[F'(x_0 \pm h_1) - F'(x_0)]/h_1$, it is contained between m and M.

(iii) *Let $f(x)$, $a \leqslant x \leqslant b$, be an integrable function, $f_1(x)$ the indefinite integral of $f(x)$, and $\varepsilon > 0$ an arbitrary number. Then there exist two functions $\varphi(x)$ and $\psi(x)$ such that* (a) $|f_1(x) - \varphi(x)| < \varepsilon$, $|f_1(x) - \psi(x)| < \varepsilon$, (b) *at every point where $f(x) \neq +\infty$ all the derivates of ψ exceed $f(x)$, and at every point where $f(x) \neq -\infty$ all the derivates of $\varphi(x)$ are less than $f(x)$.*

For the proof we refer the reader to any of the standard treatises on the Lebesgue integral [1]).

(iv) *Let $f(x)$, $a \leqslant x \leqslant b$, be an integrable function, finite except perhaps at an enumerable set E. Let $F(x)$, $a \leqslant x \leqslant b$, be a continuous function such that $\underline{D}^2 F(x) \leqslant f(x) \leqslant \overline{D}^2 F(x)$, except perhaps in E, where however F is smooth. Then F is of the form* 11.3(1).

Let $\varphi_n(x)$ and $\psi_n(x)$ be a pair of functions φ and ψ from (iii) corresponding to $\varepsilon = 1/n$, $n = 1, 2, \ldots$ Let $J[g; a, x]$ denote the integral of any function $g(t)$ over (a, x). Let $f_1(x) = J[f; a, x]$, $f_2(x) = J[f_1; a, x]$, $\Phi_n(x) = J[\varphi_n; a, x]$, $\Psi_n(x) = J[\psi_n; a, x]$. From (ii) it follows that $\underline{D}^2 \Psi_n(x) > f(x) \geqslant \underline{D}^2 F(x)$, $\overline{D}^2 \Phi_n(x) < f(x) \leqslant \overline{D}^2 F(x)$ for $x \,\overline{\in}\, E$. From this, and from the obvious inequalities

$$\underline{D}^2 \Psi_n \leqslant \overline{D}^2(\Psi_n - F) + \underline{D}^2 F, \quad \overline{D}^2 \Phi_n \geqslant \underline{D}^2(\Phi_n - F) + \overline{D}^2 F,$$

we obtain $\overline{D}^2\{\Psi_n(x) - F(x)\} \geqslant 0$ and $\underline{D}^2\{\Phi_n(x) - F(x)\} \leqslant 0$, i. e. $\overline{D}^2\{F(x) - \Phi_n(x)\} \geqslant 0$, for $x \,\overline{\in}\, E$. Using (i), we see that $\Psi_n - F$ and $F - \Phi_n$ are convex functions. Since $\varphi_n(x) \to f_1(x)$, $\psi_n(x) \to f_1(x)$, and so $\Phi_n(x) \to f_2(x)$, $\Psi_n(x) \to f_2(x)$ as $n \to \infty$, we obtain that $f_2(x) - F(x)$ and $F(x) - f_2(x)$ are convex functions. Hence $F(x) - f_2(x)$ is a linear function and the lemma follows. Incidentally, in view of (ii), the result shows that $\underline{D}^2 F(x) = \overline{D}^2 F(x) = f(x)$ almost everywhere.

(v) *If $F(x)$ is convex in an interval (a, b), then $D^2 F(x)$ exists for almost every x and is integrable over any interval $(a + \varepsilon, b - \varepsilon)$, $\varepsilon > 0$.*

Since $F(x)$ is the indefinite integral of a non-decreasing function $\xi(x)$ (§ 4.141), we have

$$1)\quad \frac{F(x+h) + F(x-h) - 2F(x)}{h^2} = \frac{1}{h^2} \int_0^h [\xi(x+t) - \xi(x-t)]\, dt.$$

[1]) See e. g. de la Vallée Poussin, *Intégrales de Lebesgue*, Saks, *Théorie de l'intégrale*.

By Lebesgue's classical theorem, $\xi'(x)$ exists almost everywhere and is integrable over $(a+\varepsilon, b-\varepsilon)$. At every point x where $\xi'(x)$ exists, we have $\xi(x+t)-\xi(x-t)=2t\,\xi'(x)+o(t)$, and so the right-hand side of (1) tends to $\xi'(x)$. This proves the lemma.

11.32. We are now in a position to prove Theorem 11.3 (iii), and even the following more general result, in which the upper and lower sums of a series with partial sums s_n mean the numbers $\overline{\lim}\, s_n$ and $\underline{\lim}\, s_n$ respectively.

If the upper sum $f^*(x)$ and the lower sum $f_*(x)$ of the series 11.1(1), *where $a_n \to 0$, $b_n \to 0$, are both integrable, and finite outside an enumerable set E of points, the series is $\mathfrak{S}[f]$, where $f(x)=\overline{D}^2 F(x)$ (or $f(x)=\underline{D}^2 F(x)$) and F is given by* 11.3(1).

For from Theorem 11.2(i') it follows that $\overline{D}^2 F(x)$ and $\underline{D}^2 F(x)$ are both integrable, and are finite for $x \,\overline{\epsilon}\, E$. The function F is smooth (§ 11.2(ii)); hence, if we put $f(x)=\overline{D}^2 F(x)$, the conditions of the last lemma but one of § 11.31 are satisfied, F is of the form 11.3(1), and this, as we know, proves the theorem.

The following remark, which requires no proof, will be useful later: *If the conditions of the last theorem are satisfied in an interval (a, b), the function $F(x)$ satisfies the equation* 11.3(1) *for $a \leqslant x \leqslant b$.*

The proof of Theorem 11.3(iii) which we have given is not very simple; it is therefore of interest to observe that Theorem 11.3(i), which is very important in itself, is much easier. For, under the hypothesis of that theorem, the function $F(x)$ satisfies the condition $D^2 F(x)=0$, and so, using Lemma 11.31(i) in its simplest form $(\underline{D}^2 F=\overline{D}^2 F=D^2 F=0)$, we obtain that the functions F and $-F$ are convex; hence F is a linear function $Ax+B$. The equation

$$(1)\qquad \tfrac{1}{4}\, a_0\, x^2 - Ax - B - \sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n^2} = 0$$

holds for all x, and so, making $x \to \infty$ and observing that the sum on the left represents a bounded function, we obtain $A=0$, $a_0=0$. Now the left-hand side of (1) is a trigonometrical series converging uniformly to 0; hence $B=a_1=b_1=a_2=\ldots=0$ and the theorem follows.

11.33. Theorem 11.32 may be generalized as follows:

If $f_*(x)$ and $f^*(x)$ are finite outside an enumerable set E, and if $f_*(x) \geqslant g(x)$, where $g(x)$ is integrable (in particular, if $f_*(x)$ is integrable), the series is a Fourier series.

In this paragraph we shall only prove the theorem in the special case $f_*(x) = f^*(x) = f(x)$[1]. The general result is a corollary of a theorem which we shall prove in § 11.6.

Let $g_1, g_2, \varphi_n, \Phi_n$ be functions which have a similar meaning to that in the proof of Theorem 9.31(iv), but correspond to the function g. It follows that, outside E, $D^2(F - \Phi_n) \geqslant \bar{D}^2 F - \bar{D}^2 \Phi_n = D^2 F - \bar{D}^2 \Phi_n \geqslant f - g \geqslant 0$. Thus $F - \Phi_n$ is convex, and, making $n \to \infty$, we obtain that $F - g_2$ is also convex. Hence $D^2(F - g_2) = f - g$ exists almost everywhere and is integrable over any finite interval (§ 11.31(v)). Thence we deduce that f is integrable, and the theorem considered follows from Theorem 11.32.

11.4. The principle of localization. It was proved in § 2.5 that the behaviour of $\mathfrak{S}[f]$ at a point x_0 depends only on the values of f in an arbitrarily small neighbourhood of x_0. This is a special case of the following more general theorem, due to Riemann, which involves arbitrary trigonometrical series with coefficients tending to 0: *The behaviour of the series* 11.1(1) *at a point x_0 depends only on the values of the function $F(x)$, defined by* 11.2(1), *in an arbitrarily small neighbourhood of x_0.* More precisely:

Let $F_1(x)$ and $F_2(x)$ be the functions F corresponding to two trigonometrical series; if $F_1(x) = F_2(x)$ in an interval $(x_0 - \varepsilon, x_0 + \varepsilon)$, or, more generally, if $F_1(x) - F_2(x)$ is equal to a linear function in $(x_0 - \varepsilon, x_0 + \varepsilon)$, the series considered are equiconvergent at the point x_0.

If two integrable functions $f_1(x)$ and $f_2(x)$ are equal in an interval $(x_0 - \varepsilon, x_0 + \varepsilon)$, then, since Fourier series may be integrated term by term, the functions F_1 and F_2, corresponding to $\mathfrak{S}[f_1]$ and $\mathfrak{S}[f_2]$, differ by a linear function in $(x_0 - \varepsilon, x_0 + \varepsilon)$; this shows that the principle of localization for Fourier series is actually a special case of the theorem just stated.

---

[1]) This result has been obtained by Banach (as a generalization of an earlier result of Steinhaus [2] for the case $g(x) = 0$) but never published.

11.41. Rajchman's theory of formal multiplication of trigonometrical series. A new approach to problems of localization is due to Rajchman, who developed the theory of formal multiplication of trigonometrical series with coefficients tending to 0 [1]). Not only does this theory enable us to obtain Riemann's results, but it can also be applied to problems where Riemann's classical method would not work.

We shall write trigonometrical series in the complex form (§ 1.43). Given two trigonometrical series

$$(1) \qquad \text{a)} \sum_{n=-\infty}^{+\infty} c_n e^{inx}, \qquad \text{b)} \sum_{n=-\infty}^{+\infty} \gamma_n e^{inx},$$

we shall call the series

$$(2) \qquad \sum_{n=-\infty}^{+\infty} C_n e^{inx}, \quad \text{where} \quad C_n = \sum_{p=-\infty}^{+\infty} c_p \gamma_{n-p},$$

their formal product, provided that the series defining C_n converge. This is certainly the case if the first of the series (1) has coefficients tending to 0 and the second converges absolutely. We shall assume for simplicity that the series considered are real, i. e. that $c_{-n} = \bar{c}_n$, $\gamma_{-n} = \bar{\gamma}_n$. It is plain that also $C_{-n} = \bar{C}_n$.

We require the following lemma, in which we suppose, as an exception to this rule, that c_n and γ_n are arbitrary complex numbers.

If $c_n \to 0$ as $n \to \pm\infty$, and if $\Sigma|\gamma_n|$ converges, then $C_n = \sum_{p=-\infty}^{\infty} c_p \gamma_{n-p}$ tends to 0 as $n \to \pm\infty$.

For let $M = \operatorname{Max}|c_n|$; then, as $n \to +\infty$,

$$|C_n| \leqslant M \sum_{p=-\infty}^{[n/2]} |\gamma_{n-p}| + \operatorname{Max}_{p>n/2} |c_p| \sum_{p=[n/2]+1}^{\infty} |\gamma_{n-p}| \leqslant$$

$$\leqslant M \sum_{q=n-[n/2]}^{\infty} |\gamma_q| + \operatorname{Max}_{p>n/2} |c_p| \sum_{q=-\infty}^{+\infty} |\gamma_q| \to 0.$$

As regards the case $n \to -\infty$, we observe that $C_{-m} = \sum_{p=-\infty}^{\infty} c'_p \gamma'_{m-p}$, where $c'_p = c_{-p}$, $\gamma'_p = \gamma_{-p}$.

If c_n and γ_n depend on a parameter, and the conditions imposed upon c_n and γ_n are satisfied uniformly, then $C_n \to 0$ uniformly.

---

[1]) Rajchman [2], [3], Zygmund [11]. In the last paper a discussion of the case of coefficients not tending to 0 is given.

11.42. We shall say that the series 11.41(1b) is *rapidly* convergent to sum s, if the series converges to s and if, moreover, $\Gamma_0+\Gamma_1+...+\Gamma_n+...<\infty$, where $\Gamma_n=|\gamma_n|+|\gamma_{n+1}|+...$ We certainly have rapid convergence if, for example, $\gamma_n=O(n^{-3})$, $n>0$. The following theorem is fundamental for the whole theory.

(i) *Suppose that* $c_n\to 0$ *and that the series* 11.41(1b) *converges rapidly to* 0 *for* x *belonging to a set* E. *Then the product* 11.41(2) *converges uniformly to* 0 *in the set* E.

Let $R_k(x)$ denote the sum of the terms $\gamma_n e^{inx}$ with $n\geqslant k$. If $x_0\in E$, $k>0$, then $|R_{-k}(x_0)|=|R_{k+1}(x_0)|\leqslant\Gamma_{k+1}$, and so the series $\sum_{-\infty}^{+\infty}|R_k(x_0)|$ is uniformly convergent in E. Now

$$S_m(x_0)=\sum_{n=-m}^{m}C_n e^{inx_0}=\sum_{n=-m}^{m}e^{inx_0}\sum_{p=-\infty}^{\infty}c_p\gamma_{n-p}=$$

$$=\sum_{p=-\infty}^{\infty}c_p e^{ipx_0}\sum_{n=-m}^{m}\gamma_{n-p}e^{i(n-p)x_0}=$$

$$=\sum_{p=-\infty}^{\infty}c_p e^{ipx_0}R_{-m-p}(x_0)-\sum_{p=-\infty}^{\infty}c_p e^{ipx_0}R_{m-p+1}(x_0).$$

Applying the lemma of the last section (with $c_p e^{ipx_0}$ and $R_{n-p}(x_0)$ instead of c_p and γ_{n-p}) we see that $S_m(x_0)$ tends uniformly to 0 for $x_0\in E$, $m\to\infty$. This proves (i).

The reader will observe that the above theorem remains true even if the coefficients c_n and γ_n of the series 11.41(1) depend themselves on the variable x, provided that the formal product is defined by 11.41(2). This is not surprising since proposition (i), as well as (ii) below, are nothing but theorems on the Laurent multiplication of arbitrary series [1]).

(ii) *If* $c_n\to 0$, *and if the series* 11.41(1b) *converges rapidly to sum* $\lambda(x)$, *the series*

$$\sum_{n=-\infty}^{\infty}C_n e^{inx} \qquad \textit{and} \qquad \lambda(x)\sum_{n=-\infty}^{\infty}c_n e^{inx}$$

are uniformly equiconvergent in the interval $(0, 2\pi)$.

---

[1]) Similar theorems can be established for other rules of multiplication, in particular for Cauchy's rule.

Let us write $\gamma_0^* = \gamma_0 - \lambda(x)$, $\gamma_n^* = \gamma_n$ for $n \neq 0$, and consider the formal product $\Sigma C_n^* e^{inx}$ of the series $\Sigma c_n e^{inx}$ and $\Sigma \gamma_n^* e^{inx}$. In view of (i) and the additional remark, the formal product converges to 0 uniformly in the interval $(0, 2\pi)$, and it is sufficient to notice that $C_n^* = C_n - \lambda(x)\, c_n$.

Now we shall state a number of corollaries which, although very simple, have important applications.

(a) *If* $\lambda(x_0) \neq 0$, *a necessary and sufficient condition that* 11.41(2) *should converge at the point* x_0, *is that* 11.41(1a) *should converge there.*

Let $\mathfrak{A}$ be any Toeplitz method of summation (§ 3.1). Observing that, if $\Sigma C_n^* e^{inx_0}$ converges to 0, it is summable $\mathfrak{A}$ to 0, we obtain:

(b) *If* $\lambda(x_0) \neq 0$, *a necessary and sufficient condition that* 11.41(2) *should be summable* $\mathfrak{A}$ *at the point* x_0 *is that* 11.41(a) *should be summable* $\mathfrak{A}$ *at that point. If the sum of the latter series is* s, *the sum of the former is* $\lambda(x_0) \cdot s$.

(c) *If the series* 11.41(a) *is uniformly convergent, or summable* $\mathfrak{A}$, *over a set* $\mathcal{E}$, *so is the series* 11.41(2). *The converse is also true if* $|\lambda(x)| \geqslant \varepsilon > 0$ *for* $x \in \mathcal{E}$.

Proposition (b) may be completed by considering limits of indetermination. Restricting ourselves to the case of ordinary convergence (the reader will have no difficulty in stating the general result) we have:

(d) *If the upper and lower sums of* 11.41(1a) *at the point* x_0 *are* $\bar{s}$ *and* $\underline{s}$ *respectively, the upper and lower sums of* 11.41(2) *are* $\lambda(x_0) \cdot \bar{s}$ *and* $\lambda(x_0) \cdot \underline{s}$ *if* $\lambda(x_0) > 0$, *and* $\lambda(x_0) \cdot \underline{s}$ *and* $\lambda(x_0) \cdot \bar{s}$ *if* $\lambda(x_0) < 0$.

11.43. Now we shall prove certain theorems about the series conjugate to formal products. It will be recalled that the series conjugate to 11.41(1a) may be obtained from the latter by replacing c_n by $c_n \varepsilon_n$, where $\varepsilon_n = -i \operatorname{sign} n$ (§ 1.13).

(i) *Under the hypotheses of Theorem* 11.42(i), *the series conjugate to the formal product converges uniformly over* E.

(ii) *Under the hypotheses of Theorem* 11.42(ii), *the series*

$$(1) \quad \text{a)} \sum_{n=-\infty}^{\infty} C_n \varepsilon_n e^{inx} \quad \textit{and} \quad \text{b)} \ \lambda(x) \sum_{n=-\infty}^{\infty} c_n \varepsilon_n e^{inx} \quad (\varepsilon_n = -i \operatorname{sign} n)$$

are uniformly equiconvergent in the wider sense.

Let $\bar{S}_n(x)$ denote the partial sums of the series (1a). Writing $c'_n = c_n e^{inx_0}$, and similarly defining C'_n and γ'_n, we have

$$\bar{S}_m(x_0) = \sum_{n=-m}^{m} \varepsilon_n C'_n = \sum_{n=-m}^{m} \varepsilon_n \sum_{p=-\infty}^{\infty} c'_p \gamma'_{n-p} = \sum_{p=-\infty}^{\infty} c'_p \sum_{n=-m}^{m} \gamma'_{n-p} \varepsilon_n =$$

$$= -i \sum_{p=-\infty}^{\infty} c'_p \sum_{n=1}^{m} (\gamma'_{n-p} - \gamma'_{-n-p}) =$$

$$= -i \sum_{p=-\infty}^{\infty} c'_p \{R_{1-p}(x_0) - R_{m-p+1}(x_0) - R_{-m-p}(x_0) + R_{-p}(x_0)\}$$

and, in view of Lemma 11.41, if $x_0 \in E$ and $m \to \infty$, the last expression tends uniformly to $-i \sum_{p=-\infty}^{\infty} c'_p \{R_{1-p}(x_0) + R_{-p}(x_0)\}$. This proves (i). To prove (ii) we use the same device as in the case of Theorem 11.42(ii). We consider the formal product $\Sigma C^*_n e^{inx}$ of the series $\Sigma c_n e^{inx}$ and $\Sigma \gamma^*_n e^{inx}$. The coefficients C^*_n depend on x, but if we define the series 'conjugate' to the product as $\Sigma \varepsilon_n C^*_n e^{inx}$, the latter series will, as the proof of (i) shows, be uniformly convergent. Since $C^*_n = c_n - \lambda(x) \gamma_n$, the theorem is established.

The following is one of the corollaries of (ii):

(a) *If the series* $\sum_{n=-\infty}^{\infty} c_n \varepsilon_n e^{inx}$ *is uniformly summable* $\mathfrak{A}$ *over a set* $\mathcal{E}$, *so is* (1a) *The converse is also true if* $|\lambda(x)| \geq \varepsilon > 0$ *over* $\mathcal{E}$.

A characteristic feature of the theorems on formal multiplication which we have proved is that we suppose next to nothing about one of the factors, whereas upon the second we impose rather stringent conditions. However, if the first series is a Fourier series, the conditions imposed upon the second series may be relaxed slightly. The reader will observe that Theorems 2.53 and 2.531 may be considered as theorems on the formal multiplication of trigonometrical series in the case when the first factor is a Fourier series.

We shall now give a number of applications of the theory of formal multiplication.

11.44. As a first application we shall show that, *given an arbitrary closed set* $E \subset (0, 2\pi)$, *there is a trigonometrical series with coefficients tending to* 0 *which converges in* E *and diverges outside* E [1]).

---

[1]) Rajchman [2]. It is plain that, if E contains one of the points $0, 2\pi$, it must contain the other.

We start with the fact that there is a trigonometrical series

$$(1)\qquad \tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)\qquad (a_n\to 0,\ b_n\to 0)$$

which diverges everywhere (§ 8.5). Let $\lambda(x)$ be a function, with Fourier coefficients $O(n^{-3})$, which is equal to 0 in E and different from 0 elsewhere [1]). The formal product of (1) by $\mathfrak{S}[\lambda]$ gives the required example, for, in view of Theorems 11.42, this product converges to 0 in E and diverges outside E.

Since, in view of Theorem 11.43(i), the series conjugate to the product considered converges in E, we obtain at once: *for every closed set E situated on the circumference of the unit circle there is a power series with coefficients tending to 0 which converges in E and diverges in the remaining points of the circumference* [2]).

11.441. The only example which we so far know of an everywhere divergent series is Kolmogoroff's example considered in § 8.5. Since that example is a Fourier series, the theory of formal multiplication was not indispensable in the argument of the previous section, and we could use Theorem 2.53 instead. Moreover, Kolmogoroff's series is fairly complicated, and it is therefore desirable to have a simpler example. Following Steinhaus, we shall show that *the series*

$$(1)\qquad \sum_{k=3}^{\infty}\frac{\cos k(x-\log\log k)}{\log k}$$

diverges for every x [3]).

Let $l_k=[\log k]$, $v_k=\log\log k$, and

$$G_n(x)=\sum_{k=n+1}^{n+l_n}\frac{\cos k(x-v_k)}{\log k},\qquad G_n=\sum_{k=n+1}^{n+l_n}\frac{1}{\log k},\qquad I_n=(v_n,v_{n+1}).$$

---

[1]) Let $\{(\alpha_n,\beta_n)\}$ be the sequence of intervals contiguous to E, and let $\lambda_n(x)$ be equal to $(x-\alpha_n)^4(\beta_n-x)^4$ in (α_n,β_n) and to 0 elsewhere. If $\eta_n>0$, $\Sigma\eta_n<\infty$, we may put $\lambda(x)=\Sigma\eta_n\lambda_n(x)$, for $\lambda'''(x)$ exists and is continuous.

[2]) For a more complete result see Mazurkiewicz [1].

[3]) Steinhaus [10]. The first example of an everywhere divergent trigonometrical series with coefficients tending to 0 was given by Steinhaus [9]. Other examples will be found in Hardy and Littlewood [9], [18]. See also Wilton [1].

Since $G_n > l_n \log(n+l_n) \to 1$, we have $G_n > 0.9$ for $n > n_0$. The inequality $|\sin u| \leqslant |u|$ gives

$$(2) \qquad 0 \leqslant G_n - G_n(x) < \frac{1}{2\log n} \sum_{k=n+1}^{n+l_n} k^2(x-v_k)^2.$$

If $n < k \leqslant n + l_n$, then $v_n < v_k \leqslant v_{n+l_n}$. Hence, if x belongs to the interval (v_n, v_{n+1}), $n \geqslant 3$, then $|x - v_k| \leqslant v_{n+l_n} - v_n$; applying the mean-value theorem we obtain $|x - v_k| \leqslant l_n/n \log n \leqslant 1/n$, and the right-hand side of (2) is less than $(n+l_n)^2\, l_n/2\, n^2 \log n < 0.6$ for $n > n_1$. Collecting the results, we see that

$$G_n(x) = G_n - (G_n - G_n(x)) > 0.9 - 0.6 = 0.3,\ x \in I_n,\ n > \mathrm{Max}\,(n_0, n_1).$$

Since every point x belongs (mod 2π) to an infinite number of the intervals I_n, the series (1) diverges for every x.

11.45. Fatou's theorem on power series. If the series

$$(1) \qquad \sum_{n=0}^{\infty} \alpha_n z^n = F(z)$$

converges at a point of the unit circle, then $\alpha_n \to 0$. The converse is false (the power series whose real part for $z = e^{ix}$ is the series 11.441(1), diverges at every point of the unit circle), but

If $\alpha_n \to 0$, the series (1) converges at every point of the unit circle where the function $f(x)$ is regular. The convergence is uniform on every closed arc of regularity.

This theorem, due substantially to Fatou [1]), is a consequence of more general results which will be established later. In view however of its importance, we shall prove it separately. Considering the real and imaginary parts of (1) for $z = e^{ix}$, we see that the theorem will be established when we have shown that, *if the series* 11.44(1) *is uniformly summable A, for $a \leqslant x \leqslant b$, to a function $g(x)$ which together with its first and second derivatives is continuous, the series is uniformly convergent in every interval (a', b') interior to (a, b).*

Let $\lambda(x)$ be a function equal to 1 in (a', b'), equal to 0 outside (a, b), and such that $\lambda'''(x)$ exists and is continuous. Since

---

[1]) Fatou [1], M. Riesz [1], [5], [6]. The part concerning uniform convergence, was first stated by M. Riesz.

the coefficients of $\mathfrak{S}[\lambda]$ are $O(n^{-3})$, the formal product of 11.44(1) by $\mathfrak{S}[\lambda]$ converges uniformly to 0 outside (a, b). By Theorem 11.42(ii), this product is uniformly summable A for $a \leqslant x \leqslant b$, to the value $\lambda(x) g(x)$. Hence it is uniformly summable A in the whole interval $0 \leqslant x \leqslant 2\pi$, to a sum $\varphi(x)$ which has a continuous second derivative. It follows that the product is $\mathfrak{S}[\varphi]$; for if α_n, β_n are the coefficients of the product, and $\varphi(r, x)$ the corresponding harmonic function, then

$$\alpha_n r^n = \frac{1}{\pi}\int_0^{2\pi} \varphi(r, x) \cos nx\, dx, \qquad \beta_n r^n = \frac{1}{\pi}\int_0^{2\pi} \varphi(r, x) \sin nx\, dx$$

and, making $r \to 1$, we see that α_n and β_n are Fourier coefficients of φ. Since $\varphi''(x)$ exists and is continuous, the numbers α_n, β_n are $O(n^{-2})$, and so $\mathfrak{S}[\varphi]$ converges uniformly. Observing that $\lambda(x) = 1$ for $a' \leqslant x \leqslant b'$, and applying Theorem 11.42(ii), we see that 11.44(1) converges uniformly over (a', b'), and the theorem is established.

The reader will notice that the condition concerning g'' was not indispensable. We only used it as a simple test ensuring the convergence of $\mathfrak{S}[\varphi]$. It would also be sufficient to assume that g satisfies the Dini-Lipschitz condition, or is continuous *and* of bounded variation.

11.46. Proof of the principle of localization. Let $\mathfrak{A}$ be a linear method of summation. We shall say that $\mathfrak{A}$ is of type U, if every trigonometrical series with coefficients tending to 0, and summable $\mathfrak{A}$ to a finite and integrable function $f(x)$, is $\mathfrak{S}[f]$. In § 11.3 we showed that ordinary convergence is of type U. It is important to notice that the method R is also of type U; this was implicitely proved in § 11.3, for the essence of the Riemann method in problems of uniqueness just consists in treating convergent series as series summable R. In § 11.6 we shall prove that Abel's method of summation is of type U.

In what follows we shall frequently consider formal products of trigonometrical series by the Fourier series of functions λ. To avoid repetition we shall tacitly assume that $\lambda''(x)$ exists and is of bounded variation. Then the Fourier coefficients of f are $O(n^{-3})$ and the theorems on formal multiplication can be applied. It will be also convenient to suppose that, if of two functions $\varphi(x)$ and $\psi(x)$ one is equal to 0 in an interval (α, β), the product $\varphi\psi$ exists and is equal to 0 in (α, β) even if the second factor is not defined in that interval.

(i) *Let $\mathfrak{A}$ be any method of summation of type U. If, for $a<x<b$, the series* 11.44(1) *is summable $\mathfrak{A}$ to a finite and integrable function $f(x)$, then, for $a' \leqslant x \leqslant b'$, the series is uniformly equiconvergent with $\mathfrak{S}[\lambda f]$, where $\lambda(x)$ is equal to 1 for $a' \leqslant x \leqslant b'$, $a<a'<b'<b$, and to 0 outside (a, b) (mod 2π). The series conjugate to* 11.44(1), *and $\bar{\mathfrak{S}}[\lambda f]$, are uniformly equiconvergent in the wider sense in the interval (a', b')* [1].

To prove the first part of the theorem we observe that the product of 11.44(1) by $\mathfrak{S}[\lambda]$ converges to 0 outside (a, b), and is summable $\mathfrak{A}$ to λf in (a, b). Hence this product is summable $\mathfrak{A}$ in the whole interval $(0, 2\pi)$ to sum $\lambda(x) f(x)$. This sum is integrable; hence the product is $\mathfrak{S}[\lambda f]$ and it remains to apply Theorem 11.42(ii). To obtain the second part of the theorem we apply Theorem 11.43(ii).

Now we are in a position to prove the Riemann principle of localization which will be established in the following general form (we preserve the notation of § 11.4):

(ii) *Let S_1 and S_2 be two trigonometrical series with coefficients tending to 0, and let $F_1(x)$ and $F_2(x)$ denote the sums of the series S_1 and S_2 integrated formally twice. If the difference $F_1(x)-F_2(x)$ is a linear function in an interval $a \leqslant x \leqslant b$, the series S_1 and S_2 are uniformly equiconvergent in every interval (a', b') interior to (a, b). The series conjugate to S_1 and S_2 are uniformly equiconvergent in the wider sense in the interval (a', b')* [2].

Let 11.44(1) be the difference of S_1 and S_2. We have to show that this series, as well as its conjugate, are uniformly convergent over (a', b'), the sum of the former being 0. Integrating 11.41(1) twice, we obtain a function $F(x) = F_1(x) - F_2(x)$ which is linear over (a, b). Since $\Delta^2 F(x, h)/h^2 = 0$ for any x interior to (a, b), and h sufficiently small, the series 11.44(1) is summable R to 0 for $a<x<b$, and it suffices to apply proposition (i).

As a special case we obtain the following theorem.

---

[1]) Rajchman [2], Zygmund [11].

[2]) Riemann [1], Rajchman [2], Neder [2], Zygmund [11].

(iii) *Suppose that the sum* $F(x)$ *of the series* 11.44(1) *integrated twice satisfies an equation*

$$(1)\qquad F(x) = Ax + B + \int_a^x dy \int_a^y f(t)\,dt, \qquad a \leqslant x \leqslant b,$$

where A *and* B *are constants, and* $f(t)$ *is a function integrable over the interval* (a, b). *Let* $f^*(x)$ *be the function equal to* $f(x)$ *in* (a, b) *and to* 0 *elsewhere* (mod 2π). *Then the series* 11.44(1) *and* $\mathfrak{S}[f^*]$ *are uniformly equiconvergent in every interval* (a', b') *interior to* (a, b). *The series conjugate to* 11.44(1), *and* $\overline{\mathfrak{S}}[f^*]$, *are uniformly equiconvergent in the wider sense in the interval* (a', b').

For the proof we notice that Fourier series may be integrated term by term; hence, if $F_1(x)$ is the sum of $\mathfrak{S}[f^*]$ integrated twice, $F_1(x)$ satisfies an equation similar to (1), and so $F(x) - F_1(x)$ is linear over (a, b).

A special case of (iii), which was already used in the proof of (ii), deserves a separate statement:

(iv) *If the sum* $F(x)$ *of the series* 11.44(1) *integrated twice is linear in an interval* (a, b), *the series* 11.44(1) *as well as its conjugate are uniformly convergent in every interval interior to* (a, b), *the sum of the former series being* 0.

11.47. Theorem 11.46(iii) states that, if $F(x)$ satisfies the equation 11.46(1), the series 11.44(1) and $\mathfrak{S}[f^*]$ are uniformly equiconvergent over (a', b'). From this and from the fact that Fourier series may be integrated term by term we deduce

Under the conditions of Theorem 11.46(iii), *the series* 11.44(1) *may be integrated formally over any interval* (a', b') *interior to* (a, b); *the series*

$$(1)\qquad \tfrac{1}{2} a_0 x + C + \sum_{n=1}^{\infty} \frac{a_n \sin nx - b_n \cos nx}{n} \qquad (C \text{ const.})$$

converges uniformly over (a', b').

It is only the second part of this theorem which needs a proof, and the result will follow when we have shown that (1) converges at some point interior to (a, b). To show this we observe that the periodic part of (1) is a Fourier series with coefficients $o(1/n)$, and so it is sufficient to apply Theorem 11.21(i).

The theorem which we have just obtained may be slightly generalized, viz., *under the same conditions as above, the series* (1) *converges uniformly, and so represents the indefinite integral of* f, *in the whole interval* $a \leqslant x \leqslant b$. This is an immediate corollary of Theorem 11.21(iii). In particular,

If the series 11.44(1) *converges in the interval* $a \leqslant x \leqslant b$, *except perhaps at an at most enumerable set* E *of points, to an integrable function* $f(x)$, *the series* (1) *converges uniformly over* (a, b) *to the integral of* f[1]).

11.48 [2]). Following Young, the series 11.44(1) is called a *restricted Fourier series*, associated with an interval (a, b) and a function $f(x) \in L(a, b)$, if this series is a formally differentiated Fourier series of a function $\Phi(x)$ which is the indefinite integral of $f(x)$ for $a < x < b$.

If 11.44(1) *is a restricted Fourier series associated with an interval* (a, b) *and a function* $f(x)$, *and if* $f^*(x)$ *has the same meaning as in* § 11.46, *the series* 11.44(1) *and* $\mathfrak{S}[f^*]$ *are uniformly equiconvergent over any interval* (a', b') *interior to* (a, b). *The series conjugate to* 11.44(1) *and* $\tilde{\mathfrak{S}}[f^*]$ *are uniformly equiconvergent for* $a' \leqslant x \leqslant b'$, *but in the wider sense.*

The theorem is a corollary of Theorem 11.46(iii) if we observe that the function $F(x)$ corresponding to 11.44(1) is of the form 11.46(1).

11.49. Riemann's formulae. Riemann deduced his principle of localization from an important formula which we shall now prove, in a slightly more general form.

Let $a < a' < b' < b$, and let $\lambda(x)$ be a function equal to 1 in (a', b'), vanishing outside (a, b) (mod 2π) and having Fourier coefficients $O(n^{-3})$.

If $F(x)$ *is the sum of the series* 11.44(1) *integrated twice, the sequences*

$$(1) \quad \tfrac{1}{2} a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) - \frac{1}{\pi} \int_a^b F(t)\, \lambda(t) \frac{d^2}{dt^2} D_n(t-x)\, dt$$

---

[1]) Lusin [2], Hobson [2]. It is sufficient to assume that the upper and lower sums of the series 11.44(1) are finite for $a \leqslant x \leqslant b$, $x \,\overline{\epsilon}\, E$, and that one of them is integrable over (a, b).

[2]) Young [15], [16]; see also Hobson's *Theory of functions*, 2, p. 686.

$$(2)\quad \sum_{k=1}^{n}(a_k \sin kx - b_k \cos kx) - \frac{1}{\pi}\int_a^b F(t)\,\lambda(t)\,\frac{d^2}{dt^2}\bar{D}_n(t-x)\,dt$$

tend uniformly to limits in the interval (a', b'). *In the case of the sequence* (1) *the limit is* 0 [1]).

In this theorem, D_n and $\bar{D}_n$ denote the Dirichlet kernel and the conjugate Dirichlet kernel respectively. Since the expressions (1) and (2) depend only on the values of $F(x)$ within the interval (a, b), the above theorem contains the principle of localization.

To grasp the meaning of the theorem suppose that $a_0 = 0$, and denote the series 11.44(1) by S; F is then a periodic function with coefficients $o(n^{-2})$. Assume for a while that the formal product of $\mathfrak{S}[F]$ and $\mathfrak{S}[\lambda]$ has coefficients $o(n^{-2})$ (which is easy to prove but is not required for the proof of the theorem). Then $F\lambda$ may be considered as the function $F_1(x)$ corresponding to a trigonometrical series S_1. Since $F(x) = F_1(x)$ in (a', b'), the series $S - S_1$ converges uniformly to 0 in every interval $(a' + \delta, b' - \delta)$, $\delta > 0$ (§ 11.46(ii)), and it suffices to observe that (1) is the difference of the n-th partial sums of the series S and S_1. Similarly we prove the part of the theorem concerning the sequence (2). In other words, Riemann's formulae are, in a degree, consequences of the principle of localization. The only defect of the above argument is that it gives convergence in the interval $(a' + \delta, b' - \delta)$ and not in (a', b'). Although this point is of minor importance, we shall prove our theorem in its complete form, first for aesthetic reasons and second since in the original paper of Riemann the interval (a', b') reduces to a point; and so the above argument could not be applied to that case [2]). We require the following lemma:

If V and W are trigonometrical series, then we have the equation $(VW)'' = V''W + 2V'W' + VW''$, *where products are formal products and dashes denote formal differentiation.*

For if c_n, γ_n, C_n denote the complex coefficients of V, W, VW respectively, the n-th coefficient of $(VW)''$ is

---

[1]) Riemann [1], Neder [2], Zygmund [11].

[2]) On the other hand, this argument imposes less stringent conditions upon λ, for, as can easily be verified, it suffices to suppose that the Fourier coefficients of λ are $o(n^{-2})$.

$$-\sum_{p=-\infty}^{\infty} c_p \gamma_{n-p} \cdot n^2$$

and it is enough to notice that $-n^2 = -(n-p)^2 + 2i(n-p)ip - p^2$.

Suppose now that $a_0 = 0$, and let S denote the series 11.44(1). The expression (1) is the n-th partial sum of the series

$$S - \mathfrak{S}''[F\lambda] = S - \{\mathfrak{S}[F]\,\mathfrak{S}[\lambda]\}'' =$$
$$= (S - \mathfrak{S}''[F]\,\mathfrak{S}[\lambda]) - 2\,\mathfrak{S}'[F]\,\mathfrak{S}'[\lambda] - \mathfrak{S}[F]\,\mathfrak{S}''[\lambda].$$

Since $\mathfrak{S}''[F] = S$ and $S - S\,\mathfrak{S}[\lambda] = S(1 - \mathfrak{S}[\lambda]) = S\,\mathfrak{S}[1-\lambda]$, we obtain the equation

$$(3) \quad S - \mathfrak{S}''[F\lambda] = S\,\mathfrak{S}[1-\lambda] - 2\,\mathfrak{S}'[F]\,\mathfrak{S}'[\lambda] - \mathfrak{S}[F]\,\mathfrak{S}''[\lambda].$$

Observing that S, $\mathfrak{S}'[F]$, $\mathfrak{S}[F]$ have coefficients tending to 0, and $\mathfrak{S}[1-\lambda]$, $\mathfrak{S}'[\lambda]$, $\mathfrak{S}''[\lambda]$ have coefficients $O(n^{-3})$ and converge to 0 in (a', b'), we see (§ 11.42) that $S - \mathfrak{S}''[F\lambda]$ converges uniformly to 0 over (a', b'). This gives the first half of the theorem. To prove the second half we notice that the series conjugate to each of the products on the right of (3) converge uniformly over (a', b') (§ 11.43), and that (2) is the n-th partial sum of the series conjugate to $S - \mathfrak{S}''[F\lambda]$.

Since the series 11.44(1) can be represented as a sum of two trigonometrical series one of which consists of the constant term $\frac{1}{2}a_0$ and the other of the remaining terms, it is sufficient to prove the theorem in the case $S = \frac{1}{2}a_0$. Integrating by parts twice, we see that (1) and (2) are equal to

$$(4) \quad \tfrac{1}{2}a_0 - \frac{1}{\pi}\int_0^{2\pi} \{F(t)\lambda(t)\}'' D_n(t-x)\,dt, \quad -\frac{1}{\pi}\int_0^{2\pi} \{F(t)\lambda(t)\}'' \tilde{D}_n(t-x)\,dt$$

respectively. Since $F(t) = {}^1/_4\, a_0 t^2$ and $\{F(t)\lambda(t)\}'' = \frac{1}{2}a_0$ for $a' \leqslant t \leqslant b'$, the simplest criteria for the convergence of Fourier series and conjugate series show that, for $a' \leqslant x \leqslant b'$, the expressions (4) tend uniformly to limits, the limit of the first being 0. This completes the proof of the theorem. We add two remarks.

(a) We supposed that $a' < b'$, but the theorem and the argument are unaffected if $a' = b'$, provided that $\lambda'(x) = \lambda''(x) = 0$ at this point. The last conditions are automatically satisfied in the whole interval (a', b') if $a' < b'$ and the Fourier coefficients of λ are $O(n^{-5})$.

(b) The first of the proofs which we have given in this section and which elucidated the meaning of the Riemann formulae shows in what sense the method of Rajchman is, in certain cases, advantageous over the original method of Riemann. Let S be the series 11.44(1). Following Rajchman, in order to remove the influence of the behaviour of S outside (a, b), we multiply S by $\mathfrak{S}[\lambda]$, where λ is a function which vanishes outside (a, b); the behaviour of $S\mathfrak{S}[\lambda]$ is known at every point. Riemann's method consists in integrating S twice, multiplying the resulting function $F(x)$ by $\lambda(x)$, and differentiating the product twice. That the resulting series S_1 is equiconvergent with S in (a', b'), is just the Riemann theorem, and it can easily be shown that S_1 converges to 0 outside (a, b). There remain two intervals, viz. (a, a') and (b', b), and Riemann's theorem tells us nothing about the behaviour of S_1 in them. Using the theorems on formal multiplication, this behaviour can be read from the formula (3), and we see that not only does this involve the series S, but also $\mathfrak{S}'[F]$, which is obtained by formal integration of S.

It must however be emphasized that the Riemann idea of introducing the function F into problems of localization is of fundamental importance. The method of formal multiplication completes it, but can in no way replace it.

11.5. Sets of uniqueness and sets of multiplicity. A point-set $E \subset (0, 2\pi)$ will be called a set of *uniqueness*, or *U*-set, if every trigonometrical series converging to 0 outside E vanishes identically. In § 11.3 we showed that every enumerable set is a *U*-set. If E is non-enumerable but does not contain any perfect subset (the existence of such sets E follows from Zermelo's Axiom) E is also a set of type *U*. This follows from the fact that the set of points where a trigonometrical series does not converge to 0 is a Borel set and so, if it does not contain a perfect subset, it must be at most enumerable[1]); this implies that the series vanishes identically. If E is a set of uniqueness, every set $E_1 \subset E$ is also a *U*-set.

A set E which is not a *U*-set will be called a set of *multiplicity*, or *M*-set. If E is of type *M*, there is a trigonometrical series which

---

[1]) See e. g. Hausdorff, *Mengenlehre*, p. 179—180.

converges to 0 outside E but does not vanish identically. Any set E of positive measure is an M-set. For let E_1, $|E_1| > 0$, be a perfect subset of E, and $f(t)$ the characteristic function of E_1. The series $\mathfrak{S}[f]$ converges to 0 at every point $x \bar{\epsilon} E$, and does not vanish identically since its constant term is $|E_1|/2\pi > 0$. It follows that it is only the case of sets of measure 0 which requires investigation, and it is a very curious fact that among perfect sets of measure 0 there exist U-sets as well as M-sets. Whether a given set E, $|E| = 0$, is of type U or of type M seems to depend on the arithmetical properties of E, and the problem of necessary and sufficient conditions — expressed in structural terms — is not yet solved.

11.51. *H*-sets are sets of uniqueness. That there exist perfect sets of type U was found independently by Mlle Nina Bary and Rajchman[1]). The latter showed that sets of type H, which we considered in § 11.1 (in particular Cantor's ternary set), are U-sets, and this result will be proved here.

Let Red $x = x - [x] =$ the non-integral part of x. We consider a sequence $\{\alpha_k\}$ of real numbers and an increasing sequence $\{n_k\}$ of positive integers. We fix a number $0 < d < 1$ and denote by E_k the set of points x where Red $\{n_k(x/2\pi) - \alpha_k\} \leqslant d$. If $E = E_1 E_2 \ldots E_k \ldots$, the set E will be called an H-set, and the reader will have no difficulty in proving, e. g. geometrically, that this definition is equivalent to that of § 11.1. It will be convenient to place the sets on the circumference of the unit circle. E_k will then consist of n_k equidistant arcs. each of length $2\pi d/n_k$. The complementary set E'_k consists of n_k intervals $I_1^{(k)}, I_2^{(k)}, \ldots, I_{n_k}^{(k)}$ of length $2\pi(1-d)/n_k$.

Let E be the set just defined and let

(1) $$\sum_{n=-\infty}^{\infty} c_n e^{inx}$$

be any trigonometrical series convergent to 0 outside E. It is convenient to suppose that this series is not necessarily real, i. e. the condition $c_{-n} = \bar{c}_n$ need not be satisfied. Let $F(x)$ be the function obtained by integrating (1) formally twice. $F(x)$ is linear in every interval I contiguous to E, and so, if the points x, $x+2h$, $x-2h$ belong

---

[1]) N. Bary [1], Rajchman [1]. Another proof, based on a different idea, will be found in Rajchman [3]. See also Verblunsky [$3_{1,4}$], Zygmund [12].

to the same interval I, the expression $\Phi_h(x) = \Delta^2 F(x, 2h)/4h^2$ is equal to 0. Take $h < 2\pi(1-d)/4$, and let x_0 be the middle-point of the interval $I_l^{(k)}$. Since the intervals $I_l^{(k)}$ are outside E, the expression $\Phi_{h/\nu}(x_0)$, where $\nu = n_k$, is equal to 0, and the same may be said of $S_{h/\nu}(x_0)$, where

$$S_{h/\nu}(x) = \frac{1}{\nu} \sum_{\mu=0}^{\nu-1} \Phi_{h/\nu}\left(x + \frac{2\pi\mu}{\nu}\right).$$

It is not difficult to see that

$$(2) \qquad S_{h/\nu}(x) = c_0 + \sum_{n=-\infty}^{\infty}{}' c_{n\nu}\, e^{in\nu x} \frac{\sin^2 nh}{n^2 h^2},$$

where the dash signifies that the term $n=0$ is omitted in summation. Since the absolute value of the sum on the right does not exceed a constant multiple of $\operatorname{Max} |c_m|$ $(m \geqslant \nu)$, we see that $S_{h/\nu}(x) \to c_0$ as $\nu \to \infty$, uniformly in x. Taking for x the point x_0 defined above, and observing that $S_{h/\nu}(x_0) = 0$, we obtain $c_0 = 0$.

To prove that $c_m = 0$, we multiply (1) by e^{-imx}. The new series converges to 0 outside E and so its constant term c_m is equal to 0. This completes the proof.

11.52. As a corollary of the previous theorem we shall show that there exist continuous functions of bounded variation with Fourier coefficients $\neq o(1/n)$ (§§ 2.213, 5.7.14). For let E denote the Cantor ternary set constructed on $(0, 2\pi)$, and $\Phi(x)$ any function continuous, of bounded variation, constant in every interval contiguous to E, but not in the whole interval $(0, 2\pi)$. *The Fourier coefficients of Φ are not $o(1/n)$.* For if they were $o(1/n)$, and if 11.44(1) denoted $\mathfrak{S}[\Phi]$ differentiated term by term, we should have $|a_n| + |b_n| = o(1)$. Since the integral of Φ is linear in the intervals contiguous to E, 11.44(1) would be summable R to 0 outside E, and so (§ 11.4) would converge to 0 outside E. Since E is a U-set, we should have $a_0 = a_1 = b_1 = \ldots = 0$, $\Phi(x) = \text{const.}$, contrary to the assumption [1]).

[1]) See also Carleman [3], Hille and Tamarkin [2].

11.53. Menchoff's example. That there are perfect M-sets of measure 0 was shown by Menchoff[1]), and is a result chronologically prior to those of § 11.51.

Consider the following set. From $E_0 = (0, 2\pi)$ we remove the interior of a concentric interval of length $|E_0|/2$. The rest E_1 consists of two intervals E_1^1 and E_1^2. From each of them we remove the interior of concentric intervals of length $|E_1^i|/3$. The rest E_2 consists of four intervals E_2^i, $i = 1, 2, 3, 4$. Having defined E_{n-1}, consisting of 2^{n-1} intervals E_{n-1}^i, we define E_n by removing the interior of intervals concentric with E_{n-1}^i and of length $|E_{n-1}^i|/(n+1)$. We put $E = E_0 E_1 E_2 \ldots$ and, following Menchoff, we shall prove that *E is a perfect M-set of measure* 0.

That E is perfect is plain. Since the measure of E_n is equal to $2\pi(1-1/2)(1-1/3)\ldots(1-1/(n+1)) = 2\pi/(n+1)$, we obtain $|E| = 0$. To prove that E is an M-set it is sufficient to construct a function $F(x)$, constant in the intervals contiguous to E, but not equivalent to a constant in $(0, 2\pi)$, which has coefficients $o(1/n)$. For $\mathfrak{S}[F]$ differentiated term by term converges to 0 outside E and does not vanish identically.

The set complementary to E_n consists of $2^n - 1$ intervals, which we shall denote by I_n^k, $k = 1, 2, \ldots, 2^n - 1$, counting from the left to the right. We define a sequence of *continuous* functions $F_1(x), F_2(x), \ldots, F_n(x), \ldots$ $(0 \leqslant x \leqslant 2\pi)$ satisfying the following conditions (i) $F_n(0) = F_n(2\pi) = 0$, $F_n(\pi) = 1$, (ii) $F_n(x)$ is constant in the intervals I_n^k, $k = 1, 2, \ldots, 2^n - 1$, and linear in the intervals E_n^i, $i = 1, 2, \ldots, 2^n$, (iii) $F_{n+1}(x) = F_n(x)$ in every I_n^k. Moreover, we suppose that (iv) if I_{n+1}^k is contained in an interval E_n^i, the value of $F_{n+1}(x)$ in I_{n+1}^k is equal to the mean value of F_n at the end-points of E_n^i. These conditions determine the functions $F_n(x)$ uniquely (we leave it to the reader to draw the graphs of the curves). It is easy to verify that $|F_n'(x)| \leqslant (n+1)/\pi$, $|F_{n+1}(x) - F_n(x)| \leqslant 1/2^n(n+2)$. It follows that the sequence $\{F_n(x)\}$ converges uniformly to a continuous function $F(x)$, and that $|F(x) - F_n(x)| < 1/n\, 2^{n-1}$.

Let C_n be the complex Fourier coefficients of $F(x)$. To show that $nC_n = o(1)$, we write

---

[1]) Menchoff [1]; see also N. Bary [1], Rajchman [4], Zygmund [13]. In the last paper it is shown that, if $n_{k+1}/n_k > \lambda > 3$, $a_k \to 0$, $\Sigma a_k^2 = \infty$, the product $\prod_{k=1}^{\infty}(1 + a_k \cos n_k x)$ may be written in the form of a trigonometrical series, which converges to 0 almost everywhere (but not everywhere).

$$(1)\quad \begin{aligned} n\int_0^{2\pi} F e^{-inx}\,dx &= n\int_0^{2\pi}(F-F_N)\,e^{-inx}\,dx + n\int_0^{2\pi} F_N e^{-inx}\,dx = \\ &= n\int_0^{2\pi}(F-F_N)\,e^{-inx}\,dx - i\int_0^{2\pi} F'_N e^{-inx}\,dx = A+B, \end{aligned}$$

where n and N are positive. Since $F(x) = F_N(x)$ outside E_N, $|A|$ does not exceed $n\,|E_N|\cdot\text{Max}\,|F-F_N| < 2\pi n/N^2 2^{N-1} = O(1/\log^2 n)$, if N is defined by the condition $2^{N-1} \leqslant n < 2^N$. Passing to the integral B, we observe that $F'_N(x)$ is equal to $\pm(N+1)/\pi$ in E_N and to 0 elsewhere. To estimate the integral of e^{-inx} over any interval belonging to E_N we have two inequalities: the absolute value of the integral exceeds neither the length of the interval nor $2/n$. The first inequality is more advantageous for intervals not large in comparison with $1/n$, the second for larger intervals. However, neither of these two inequalities alone would enable us to show that $B = o(1)$, and to overcome the difficulty we proceed as follows.

Let $\nu = \nu_N < N$ be a positive integer which we shall define presently; hence $E_N \subset E_\nu$. We write $F'_N(x) = g_N(x) + h_N(x)$, where $g_N(x)$ vanishes outside E_ν and is equal to $\pm(N+1)/\pi$ in E_ν; the sign '+' corresponds to the interval $(0, \pi)$, the sign '—' to $(\pi, 2\pi)$. Then

$$B = -i\int_0^{2\pi} g_N(x)\,e^{-inx}\,dx - i\int_0^{2\pi} h_N(x)\,e^{-inx}\,dx = B' + B'',$$

$$|B'| \leqslant 2^\nu\left(\frac{2}{n}\right)\frac{N+1}{\pi}, \qquad |B''| \leqslant |E_\nu - E_N|\,\frac{(N+1)}{\pi} = \frac{2\,(N-\nu)}{\nu+1},$$

since g_N vanishes outside E_ν, h_N vanishes outside $E_\nu - E_N$, and both $|g_N|$ and $|h_N|$ do not exceed $(N+1)/\pi$. If we put $\nu = N - [\sqrt{N}]$, we obtain $B'' = O(N^{-1/2}) = O(\log^{-1/2} n)$, $B' = O(N 2^{-\sqrt{N}}) = O(\log^{-1/2} n)$ and, collecting the results, $nC_n = O(\log^{-1/2} n) = o(1)$.

11.54. If E_1 and E_2 are sets of uniqueness, their sum $E_1 + E_2$ may be a set of multiplicity. We obtain an example by breaking up the interval $(0, 2\pi)$ into two sets E_1 and E_2, each without a perfect subset. Although E_1 and E_2 are U-sets (§ 11.5), their sum is not. This example may be not entirely convincing and it is natural to ask whether the situation is the same if we restrict ourselves to the domain of Borel sets. The answer to this problem is not known. In the case of closed sets we have the following theorem due to Mlle Bary [1]).

---

[1]) N. Bary [1].

If $E_1, E_2, \ldots, E_n, \ldots$ are closed U-sets, their sum $E = E_1 + E_2 + \ldots$ is a U-set.

We shall require the following lemma:

Let $\mathcal{E}$ be a closed set of uniqueness and J an open interval. If a trigonometrical series S with coefficients tending to 0 (i) *converges to* 0 *almost everywhere in J,* (ii) *has partial sums bounded at every point of $J - \mathcal{E}$, the series converges to* 0 *at every point of J.*

We may suppose that $J\mathcal{E} \neq 0$, for otherwise the lemma follows from Theorem 11.46(iii) and the remark of § 11.32. Now let δ be any interval contained in J and without points in common with $\mathcal{E}$. Since S converges to 0 almost everywhere in δ, and has partial sums bounded at every point of δ, S converges to 0 everywhere in δ. Hence S converges to 0 in $J - \mathcal{E}$. Let $\lambda(x)$ be a function vanishing outside J and positive in J. The formal product S_1 of S by $\mathfrak{S}[\lambda]$ converges to 0 outside J and in the set $J - \mathcal{E}$. Since $\mathcal{E}$ is a U-set, S_1 converges to 0 everywhere. Taking into account that $\lambda(x) > 0$ in J, we see that S converges to 0 in J, and the lemma is established.

Suppose now that there is a trigonometrical series S with coefficients tending to 0, converging to 0 outside E, but not everywhere; let R be the set of points at which the partial sums $s_n(x)$ of S are unbounded. R is a product of open sets, for if G_N denotes the set of points where at least one of the functions $|s_n(x)|$ exceeds N, then G_N is an open set and $R = G_1 G_2 \ldots G_N \ldots$ The set R is contained in E; outside E the series converges to 0. Since $|E| = 0$ and S is not identically equal to 0, it follows (§ 11.32) that $R \neq 0$. We may write $R = RE_1 + RE_2 + \ldots$, and since sets which are products of open sets are not of the first category in themselves [1], there is an n_0 such that RE_{n_0} is not non-dense in R. In other words, there is an open interval J such that $JR \neq 0$ and JRE_{n_0} is dense in JR. From this and from the fact that E_{n_0} is closed, we deduce that $JRE_{n_0} \supset JR$, i. e. $JRE_{n_0} = JR$. We write $E_{n_0} = \mathcal{E}$ and apply the lemma. The series S converges to 0 almost everywhere in J and has partial sums bounded at every point of the set $J - JR = J - JR\mathcal{E} \supset J - \mathcal{E}$. Hence S converges to 0 everywhere in J, contrary to the result $JR \neq 0$ obtained previously. This proves the theorem.

[1] See e. g. Hausdorff, *Mengenlehre*, 142 (Satz XI).

11.6. Uniqueness in the case of summable trigonometrical series. In § 11.3 we obtained a number of theorems on the uniqueness of the representation of a function by means of a convergent trigonometrical series. Since however there exist functions whose Fourier series diverge everywhere, it is natural to ask for theorems of uniqueness for summable trigonometrical series. We shall restrict ourselves to Abel's method of summation which has an important function-theoretic significance. Since Abel's method applies to series with coefficients not tending to 0, we begin by investigating what conditions must we impose upon the coefficients of the series considered.

Of the two series

$$(1) \qquad \text{a)} \ \sum_{n=1}^{\infty} n \sin nx, \qquad \text{b)} \ \tfrac{1}{2} + \sum_{n=1}^{\infty} \cos nx,$$

the first is summable A to 0 for every x; the second for every $x \neq 0 \pmod{2\pi}$. This shows that: (*a*) for series

$$(2) \qquad \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

summable A and having coefficients $\neq o(n)$, the theorem of uniqueness is false, (*b*) if we drop the condition $a_n \to 0$, $b_n \to 0$, we cannot introduce sets of uniqueness such as the set E of Theorem 11.32.

We write

$$f(r, x) = \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) r^n,$$

$$f_*(x) = \varliminf_{r \to 1} f(r, x), \qquad f^*(x) = \varlimsup_{r \to 1} f(r, x).$$

The functions $f^*(x)$ and $f_*(x)$ may be called the upper and lower Abel sums of the series (2). We shall prove the following two theorems, the second of which is a very special case of the first.

(i) *If the functions $f_*(x)$ and $f^*(x)$ corresponding to the series* (2) *with coefficients $o(n)$ are both finite everywhere, and if $f_*(x) \geqslant \chi(x)$, where χ is integrable,* (2) *is a Fourier series.*

(ii) *If the series* (2) *with coefficients $o(n)$ is, for every x, summable A to 0, then $a_0 = a_1 = b_1 = \dots = 0$.*

In the case of coefficients tending to 0, propositions (i) and (ii) were established by Rajchman [1]). His method applies, without essential changes, to a slightly more general case, viz. when the periodic part of the series (2) integrated twice is the Fourier series of a continuous function [2]); in particular when we have $|a_n|+|b_n|=O(n^{1-\eta}), \eta>0$. The proof of propositions (i) and (ii), as they are stated, requires new devices, and this final step was taken by Verblunsky [3]).

The proof of (i) will be based on a number of lemmas. It will not impair the generality if from the start we assume that $a_0=0$.

11.601. Rajchman's inequalities. These are fundamental for the whole argument and may be stated as follows. *If*

$$C-\sum_{n=1}^{\infty}\frac{a_n\cos nx+b_n\sin nx}{n^2} \tag{1}$$

is the Fourier series of a function $F(x)$, *and if* $f^*(x)$ *and* $f_*(x)$ *are the upper and lower Abel sums of the series* (1) *differentiated twice, then, at every point* x_0 *where* (1) *is summable A, the intervals* $(\underline{D}^2 F(x_0), \overline{D}^2 F(x_0))$ *and* $(f_*(x_0), f^*(x_0))$ *have points in common, i. e.*

$$\underline{D}^2 F(x_0)\leqslant f^*(x_0), \qquad f_*(x_0)\leqslant \overline{D}^2 F(x_0)\ [4]). \tag{2}$$

Let $x_0=0$ and let $F(r,x)$ be the harmonic function corresponding to the series (1). We may assume that $F(0)=0$, i. e. that $F_r=F(r,0)\to 0$ as $r\to 0$. To prove the first inequality (2), it is sufficient to show that, for any m, the inequality $\underline{D}^2F>m$ implies $f^*\geqslant m$. We may also assume that $m=0$, for otherwise we may consider $F(x)-m(1-\cos x)$ instead of $F(x)$. Suppose, contrary to what we want to prove, that $f^*(0)<0$. From the Laplace equation

$$\frac{1}{r^2}\frac{\partial^2 F(r,x)}{\partial x^2}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial F(r,x)}{\partial r}\right)=0$$

---

1) Rajchman [5].

2) see e. g. Zygmund [14]; M. Riesz [7] was the first to consider problems of uniqueness in the case of coefficients not tending to 0.

3) Verblunsky [3_2].

4) Rajchman [5]; Rajchman and Zygmund [1], Verblunsky [3_1]. It can be shown that, if $D^2F(x_0)$ exists and is finite, then $f_*(x_0)=f^*(x_0)=D^2F(x_0)$ (Fatou [1]), but, in the general case, the interval (f_*, f^*) need not be contained in $(\underline{D}^2 F, \overline{D}^2 F)$; see Rajchman and Zygmund [1].

we obtain that rF_r', where the dash denotes differentiation with respect to r, is an increasing function of r in an interval $r_0 \leqslant r < 1$. Since $F_r \to 0$ as $r \to 1$, the mean-value theorem gives $F_r/\log r = \rho F_\rho'$, $r_0 \leqslant r < \rho < 1$, and hence, for a σ contained in $(\rho, 1)$,

$$F_r/\log r - F_\rho/\log \rho = \rho F_\rho' - \sigma F_\sigma' < 0.$$

To show that this is impossible, it is enough to prove that $\overline{\lim_{r\to1}} \dfrac{d}{dr}\left\{\dfrac{F_r}{\log r}\right\} < 0$. Let $\Delta = \Delta(r,t) = 1 - 2r\cos t + r^2$, $P_r(t) = \frac{1}{2}(1-r^2)/\Delta$, $\varphi(t) = \{F(t) + F(-t) - 2F(0)\}/\sin^2 t$. From Poisson's formula we obtain

$$\underline{\lim_{r\to1}} \frac{d}{dr}\left\{\frac{rF_r}{1-r^2}\right\} = \underline{\lim_{r\to1}} \frac{1}{\pi}\int_0^\pi [F(t) + F(-t)]\frac{1-r^2}{\Delta^2}dt = \tag{3}$$

$$= \underline{\lim_{r\to1}} \frac{1}{\pi}\int_0^\eta \varphi(t)\sin^2 t \frac{1-r^2}{\Delta^2}dt = \underline{\lim_{r\to1}}\left\{-\frac{1}{\pi r}\int_0^\eta \varphi(t)\sin t \frac{d}{dt}P_r(t)\,dt\right\},$$

where η, $0 < \eta \leqslant \pi$, is any fixed number. Taking η so small that $\varphi(t) > h > 0$ for $0 < t \leqslant \eta$, replacing $\varphi(t)$ by h, and integrating by parts, we find that the right-hand side of (3) exceeds

$$\frac{h}{\pi}\lim_{r\to1}\int_0^\eta \cos t\, P_r(t)\,dt = \frac{h}{\pi}\lim_{r\to1}\int_0^\pi \cos t\, P_r(t)\,dt = \tfrac{1}{2}h > 0.$$

Now, if $c(r) = (1-r^2)/r\log r$, we have

$$\left(\frac{F_r}{\log r}\right)' = c(r)\left(\frac{rF_r}{1-r^2}\right)' + c'(r)\frac{rF_r}{1-r^2}. \tag{4}$$

Since $c(r) \to -2$, $c'(r) = O(1-r)$, the upper limit, for $r \to 1$, of $(F_r/\log r)'$ is negative, and the first inequality of (2) follows. Applying this inequality to $-F(x)$, we obtain the remaining inequality.

11.602. If P is a linear set of points, we shall call a *portion* of P, any non-empty product of P by an open interval I.

Let P be a perfect set and $\{f_n(x)\}$, $n = 1, 2, \ldots$, a sequence of continuous functions defined in P and bounded at every point of P. Then there is a portion Π of P in which the sequence $\{f_n(x)\}$ is uniformly bounded.

Let $E_{n,m}$ $(m, n = 1, 2, \ldots)$ be the set of points where $|f_n(x)| \leqslant m$, and let $H_m = E_{1,m}E_{2,m}E_{3,m}\ldots$ The sets $E_{n,m}$, and so also the sets H_m, are closed. Since P is the sum of all H_m, at least one of the terms, say H_{m_0}, is not non-dense over P, i. e. is dense in a portion Π of P. Being closed, it contains Π. Hence $|f_n(x)| \leqslant m_0$ for $n \geqslant 1$, $x \in \Pi$, and the lemma follows.

11.603. A function $g(x)$ is said to be upper semi continuous if, for every sequence $\{x_n\} \to x$, we have $\lim_{n\to\infty} g(x_n) \leqslant g(x)$. An important property of an upper semi-continuous function is that it attains its maximum in every finite interval; the proof is immediate.

If $\Phi(x)$ is an upper semi-continuous function satisfying the inequality $\overline{D}^2\Phi \geqslant 0$, the function Φ is convex.

The proof is a mere repetition of the argument of § 11.31(i) (with $E = 0$).

11.604. *Let $\gamma_2(x)$ denote the second integral of $\gamma(x)$. If, under the hypotheses of Theorem* 11.6(i), *the series* 11.601(1) *is, for $a<x<b$, summable A to a continuous or, more generally, upper semi-continuous function $F(x)$, the difference $F(x)-\gamma_2(x)$ is convex for $a<x<b$.*

Taking account of the preceding lemma, the proof is contained in the proof of Theorem 11.31(iv) where we showed that, with the notation of that paragraph, $F(x)-f_2(x)$ was convex; it is sufficient to observe that, in view of Lemma 11.601, we have $\gamma(x) \leqslant \overline{D}^2F(x)$.

The last lemma we shall require is

11.605. *If the series $u_0+u_1+u_2+\dots$ has Abel's upper and lower sums finite, the series $u_1+u_2/2+u_3/3+\dots$ is summable A.*

For if $g(r)=u_0+u_1r+\dots$, then $G(r)=\sum_{n=1}^{\infty}\frac{u_n}{n}r^n=\int_0^r\frac{g(\rho)-u_0}{\rho}d\rho$. Since the integrand is bounded, we have $|G(r)-G(r')| \to 0$ as $r\to 1$, $r'\to 1$, and the lemma follows.

Suppose that the u_n are functions of a parameter x. *If the function $g(r)$ is uniformly bounded for $0 \leqslant r < 1$ and x belonging to a set E, then the series $u_1+\frac{1}{2}u_2+\dots$ is uniformly summable A for $x \in E$.*

11.606. We now pass on to the proof of Theorem 11.6(i). Applying Lemma 11.605 twice to the series 11.6(2), we see that

(1) $$-\sum_{n=1}^{\infty}\frac{a_n\cos nx+b_n\sin nx}{n^2}$$

is summable A for every x (using well-known Tauber's theorem that series summable A and having coefficients $o(1/n)$ are convergent[1]), we see that (1) converges for every x; this result will not be required in the proof). The main point of the proof will be to show that the sum $F(x)$ of (1) is continuous, a result which is immediate if e. g. $a_n \to 0$, $b_n \to 0$. Let

$$p_1(r,x) = -\sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n} r^n, \quad p_2(r,x) = -\sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n^2} r^n.$$

We begin by proving that in every perfect set P there is a portion Π, such that $p_1(r,x)$ is bounded for $0 \leqslant r < 1$, $x \in \Pi$. For, if $r_1 < r_2 < \ldots$ is a sequence tending to 1 sufficiently slowly, then $|p_1(r_n, x) - p_1(r, x)| \leqslant 1$, for $r_{n-1} \leqslant r \leqslant r_n$, $0 \leqslant x \leqslant 2\pi$. In view of Lemma 11.605, $\lim p_1(r_n, x)$ exists for every x. Since the sequence $p_1(r_n, x)$ is uniformly bounded in a portion Π of P (§ 11.602), the same may be said of the expression $p_1(r, x)$.

From this and the last remark of § 11.605, we see that, in every perfect set P, there is a portion Π in which the function $F(x) = \lim p_2(r, x)$ is continuous. In particular, taking $P = (0, 2\pi)$, we obtain that the set Δ of discontinuities of F is nowhere dense in $(0, 2\pi)$.

Suppose, contrary to what we want to prove, that $\Delta \neq 0$. First of all, Δ cannot contain isolated points. For, if x_0 were one, consider the difference $\delta(x) = F(x) - \chi_2(x)$ in the neighbourhood of x_0. Since $\delta(x)$ is convex to the right and to the left of x_0 (§ 11.604), the limits $\delta(x_0 \pm 0)$ exist, and so, in view of Theorem 11.21(ii), $\delta(x_0 + 0) = \delta(x_0 - 0) = \delta(x_0)$. Hence $\delta(x)$ is continuous at x_0, and so is $F(x)$.

Δ being dense in itself, the set $\bar{\Delta}$ of limiting points of Δ is perfect. If (α, β) is any interval contiguous to $\bar{\Delta}$, the function $\delta(x)$ is convex for $\alpha < x < \beta$, and $\delta(\alpha + 0) = \delta(\alpha)$, $\delta(\beta - 0) = \delta(\beta)$. Let $\Pi = J\bar{\Delta}$ be a portion of $\bar{\Delta}$ in which $F(x)$, and so also $\delta(x)$, is continuous; J denotes an open interval. Being convex in any interval belonging to $J - \Pi$, the function $\delta(x)$ is upper semi-continuous in J. The same may be said of $F(x)$. Applying Lemma 11.604, we obtain that $\delta(x)$ is convex, and so also continuous, in J. This shows that F is continuous in J. Hence $\Delta = 0$, i. e. F is everywhere continuous.

---

[1]) See e. g. Landau, *Darstellung und Begründung*.

By Lemma 11.604, the difference $\delta(x)=F(x)-\chi_2(x)$ is convex over $(-\infty, \infty)$. To complete the proof of the theorem, we observe that $D^2F(x) = D^2\chi_2(x) + D^2\delta(x) = \chi(x) + D^2\delta(x)$ exists for almost every x and is integrable (§ 11.31(ii), (v)). Let $f(x) = \mathrm{Max}\{f_*(x), D^2F(x)\}$. Using Lemma 11.601, we see that $f(x)$, which is contained between $f_*(x)$ and $f^*(x)$, is everywhere finite and satisfies the inequality $\underline{D}^2F(x) \leqslant f(x) \leqslant \bar{D}^2F(x)$. By Lemma 11.31(iv), $F(x)$ is of the form 11.3(1); this, as we know, proves that 11.6(2) is $\mathfrak{S}[f]$, and the theorem is established. Incidentally we obtain that, under the conditions of Theorem 11.6(i), $f_*(x) = f^*(x)$ for almost every x.

11.61 [1]. *If the conditions of Theorem* 11.6(i) *are satisfied, except that $f_*(x)$ and $f^*(x)$ may be infinite at a finite number of points $x_1, x_2, \dots, x_k$, the series* 11.6(2) *differs from a Fourier series by a linear combination of the series $D(x-x_i)$, $i=1,2,\dots,k$, where $D(x)$ denotes the second series* 11.6(1).

We may again assume that $a_0=0$. Repeating the proof of Theorem 11.6(i), we obtain that $F(x)$ is everywhere continuous and that, in each of the intervals (x_{i-1}, x_i), $F(x)$ is of the form 11.3(1), with A and B depending on i. The points x_i may be angular points for the function $F(x)$. Let $D_1(x)$ denote the series $\cos x + \cos 2x + \dots$ The sum of the series $D_1(x)$ integrated twice has an angular point for $x=0$ and nowhere else (mod 2π). Therefore, if we substract from 11.6(2) a linear combination of the series $D_1(x-x_i)$, the function F corresponding to the difference has no angular points, i. e. we shall have the formula 11.3(1) with A and B constant throughout the interval $(0, 2\pi)$. It follows that the difference considered is a Fourier series, and the theorem is established. As a corollary we obtain that, if the series 11.6(2), with $|a_n| + |b_n| = o(n)$, is summable A to 0 for $x \neq x_0$, the series is a constant multiple of $D(x-x_0)$.

11.62. *Theorem* 11.6(i) *holds even if the functions $f_*(x)$ or $f^*(x)$, or both, are infinite in a set E, provided that E is at most enumerable and that $F(x)$ is smooth in E.* It is important to observe that *the latter condition is certainly satisfied when* $|a_n| + |b_n| \to 0$. The proof may be left to the reader, since it is wholly similar to that of Theorem 11.6(i), if the lemmas of § 11.31 are used in their complete form.

---

[1]) Verblunsky [3_2]; cf. also Zygmund [14].

There are other generalizations of Theorem 11.6(i). The reader interested in the subject will find them in the papers quoted. Here we will only mention one of these generalizations, viz. that *all the theorems of uniqueness established in this chapter hold if integration is understood in the Denjoy-Perron sense* [1]). This is due to the fact that all the lemmas on which our proofs are based hold for the Denjoy-Perron integral. Similarly, the Denjoy-Perron integral may be introduced into theorems on localization. For example, Theorem 11.46(iii) remains true in the new case.

11.7. Miscellaneous theorems and examples.

1. Show that Steinhaus's theorem, i. e. that

$$\overline{\lim_{n\to\infty}}\, |a_n \cos nx + b_n \sin nx| = \overline{\lim_{n\to\infty}} \sqrt{a_n^2 + b_n^2},$$

except in a set of mesure 0, can be proved by the method of § 11.11.

[Observe that, if m is a positive integer, E an arbitrary set of positive measure, and $n_k \to \infty$, then

$$(1) \qquad \int\limits_E \cos^{2m}(n_k x + \alpha_{n_k})\, dx \to |E| \binom{2m}{m} 2^{-2m},$$

and that, for m large, the right-hand side of (1) is of order $m^{-1/2}$].

2. Theorem 11.21(i) remains true if a_n and b_n are $O(1/n)$. Hardy and Littlewood [20].

[Supposing that 11.1(1) converges to 0, we write

$$\frac{F(x+t) - F(x-t)}{2t} = \sum_{n=1}^{\infty} A_n(x) \frac{\sin nh}{nh} = \sum_{n=1}^{kN} + \sum_{n=kN+1}^{\infty} = P_h + Q_h,$$

where $N = [1/h]$, and $k > 0$ is an integer. If k is large, then Q_h is small. Hence 11.1(1) is summable L to 0. Conversely, if that series is summable L, it is summable $(C, 2)$ (§ 3.5) and, as the argument of § 11.21 shows, its partial sums are bounded. Hence it is summable $(C, 1)$ (§ 10.44) and it is sufficient to apply the Hardy theorem of § 3.23].

Abel's transformation shows that, for fixed k, $P_h \to 0$ with h.

3. Suppose that $|a_n| + |b_n| = O(1/n)$, so that 11.1(1) is the Fourier series of a function $f(x)$. A necessary and sufficient condition for the convergence, at a point x, of the series conjugate to 11.1(1), is the convergence of the integral

---

[1]) Besides the papers quoted, see also P. Nalli [1].

$$-\frac{1}{\pi}\int_0^{\pi}\frac{f(x+t)-f(x-t)}{2\operatorname{tg}\frac{1}{2}t}\,dt,$$

which represents then the sum of the conjugate series. Hardy and Littlewood [20].

4. Let the series 11.1(1) be summable A, for $a<x<b$, to a non-negative function $f(x)$. A necessary and sufficient condition that the function $f(x)$ should be integrable over (a,b), is that the series

$$(*)\qquad \tfrac{1}{2}a_0x+\sum_{n=1}^{\infty}(a_n\sin nx-b_n\cos nx)/n$$

should converge for $x=a$ and $x=b$. Verblunsky [4].

[Let $F(x)$ be the sum of (*). $F(x)$ is monotonic in the interior of (a,b), and $f\in L(a,b)$ if and only if $F(a+0)$ and $F(b-0)$ are finite. Since the coefficients of (*) are $o(1/n)$, it is sufficient to apply Theorem 11.21(ii)].

5. Let S_1 and S_2 be two trigonometrical series with coefficients $o(1/n)$ and $O(1/n)$ respectively. If S_i converges to s_i, $i=1,2$, at a point x, the formal product of S_1 and S_2 converges to s_1s_2 at that point.

As the example $S_1=S_2=\Sigma n^{-1}\sin nx$, $x=0$, shows, the theorem is not true if both factors have coefficients $O(1/n)$.

6. (i) If the sine expansion of a function $f(x)$, $0<x<\pi$, has coefficients $o(1/n)$, the cosine expansion of $f(x)$ converges at the point $x=0$ and has the sum 0. (ii) If the sine expansion of $f(x)$ has coefficients $o(1/n)$ and converges uniformly in the neighbourhood of $x=0$, the cosine expansion of $f(x)$ also converges uniformly in the neighbourhood of $x=0$. (iii) In the previous theorem the rôle of sine and cosine series may be interchanged, provided that $f(0)=0$.

[To prove (i), consider the product of the sine development of f by the Fourier series of the function $\operatorname{sign} x$, $|x|<\pi$].

7. Given a function $F(x)$, we write

$$\Delta^k F(x,2h)=\sum_{j=0}^{k}\binom{k}{j}F(x+(k-2j)h),$$

Let $F(x)$ be the sum of the series 11.6(2) integrated term by term k times. We shall say that 11.6(2) is summable, at the point x, by the k-th method of Riemann, or summable R_k, to sum s, if the function F exists in the neighbourhood of x, and if

$$(**)\quad \lim_{h\to 0}\frac{\Delta^k F(x,2h)}{(2h)^k}=\lim_{h\to 0}\left[\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)\left(\frac{\sin nh}{nh}\right)^k\right]=s.$$

If $|a_n|+|b_n|=o(n^{\alpha})$, $\alpha>-1$, and if the series 11.6(2) is summable (C,α) at the point x, the series is also summable R_k, $k>\alpha+1$, to the same sum. Kogbetliantz [2], Verblunsky [5].

[A consequence of Theorem 10.5.10].

8. If $|a_n|+|b_n|=o(n^r)$, $r=0,1,2,\ldots$, $k=r+1$, and if 11.6(2) is summable (C,r), at the point x, to sum s, we still have the relation (**), where h tends to 0 through a set of points having 0 as a point of density.

See Rajchman and Zygmund [2]. In the same way we can generalize Theorem 10.42.

9. A sequence $\{a_n\}$ is said to be summable R_2' to the limit s, if the expression

$$\frac{2}{\pi}\sum_{n=1}^{\infty} a_n \frac{\sin^2 nh}{n^2 h}$$

converges in the neighbourhood of $h=0$ and tends to 0 as $h \to 0$. Show that, if $\{a_n\}$ converges, it is also summable R_2' to the same limit.

[See § 1.8.3; the theorem is practically identical with Theorem 11.2(ii)].

10. The methods R_2 and R_2' are not comparable. See Marcinkiewicz [2].

11. The conditions imposed upon the Fourier coefficients of the function $\lambda(x)$ of Theorem 11.49 are unnecessarily stringent; it is sufficient to suppose that $\lambda''(x)$ is continuous and of bounded variation.

[Consider the formula 11.49(3) and use Theorem 2.531. It is also sufficient to suppose that $\lambda'' \in \mathrm{Lip}\,\alpha$, $\alpha>0$].

12. Let the series 11.44(1) have coefficients $o(n^\alpha)$, $\alpha>-1$, and let k be any integer $>\alpha+\frac{1}{2}$. If $F(x)$ denotes the sum of the series 11.44(1) integrated term by term k times, and if $\lambda(x)$ is a function which is equal to 0 outside (a,b), equal to 1 in (a',b'), $a<a'<b'<b$, and has a sufficient number of derivatives, the differences

$$\sum_{k=0}^{n} A_k(x) - \frac{(-1)^k}{\pi}\int_a^b F(t)\,\lambda(t)\,\frac{d^k}{dt^k}D_n(t-x)\,dt,$$

$$\sum_{k=1}^{n} B_k(x) - \frac{(-1)^k}{\pi}\int_a^b F(t)\,\lambda(t)\,\frac{d^k}{dt^k}\bar{D}_n(t-x)\,dt,$$

are uniformly summable (C,α) over (a',b'), the limit of the first being 0.

See Zygmund [11], where the second expression is written in a slightly different form.

13. Let S be any trigonometrical series with coefficients tending to 0, and let $f^*(x)$ and $f_*(x)$ be the upper and lower Abel sums of S. If f^* is integrable, and if f_* and f^* are finite outside a closed set E of measure 0, the difference $S-\mathfrak{S}[f^*]$ converges to 0 outside E. If, in particular, E is a U-set, then $S=\mathfrak{S}[f^*]$.

CHAPTER XII.

Fourier's integrals.

12.1. Fourier's single integral. Given a function $f(x)$, $-\infty < x < \infty$, consider the expression

$$(1) \quad S_\omega(x) = S_\omega(x; f) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \frac{\sin \omega (x-t)}{x-t} \, dt, \quad \omega > 0.$$

This integral exists if $|f(t)|/(1+|t|)$ is integrable over $(-\infty, \infty)$, and so in particular if $f \epsilon L(-\infty, \infty)$, or, using Hölder's inequality, if $f \epsilon L^r(-\infty, \infty)$, $r > 1$. It is an important fact that, if $f(x)$ satisfies conditions ensuring the convergence of Fourier series, then $S_\omega(x) \to f(x)$ as $\omega \to \infty$. This result is known as Fourier's representation of a function by means of a single integral, and is a consequence of the results established in Chapter II and of the following theorem:

Let us fix an arbitrary interval $J_a = (a, a + 2\pi)$, and let $f_a(x)$ be the function of period 2π, which is equal to $f(x)$ in J_a. We suppose that $|f(x)|/(1+|x|) \epsilon L(-\infty, \infty)$. Let $s_n(x) = s_n(x; f_a)$ be the partial sums of $\mathfrak{S}[f_a]$. Then, for x belonging to any interval J'_a interior to J_a, the difference $S_\omega(x) - s_{[\omega]}(x)$ tends uniformly to 0 as $\omega \to \infty$ [1]).

12.11. We base the proof of the theorem on a number of lemmas. First of all, it is sufficient to consider, instead of $S_\omega - s_{[\omega]}$, the difference $S_\omega - s^*_{[\omega]}$, where s^*_n are the modified partial sums (§ 2.3). To fix ideas, we assume that $a=0$, and write $J_0, J'_0, f_0(x)$ instead of $J_a, J'_a, f_a(x)$.

[1]) See Hobson [3], Pringsheim [2].

(i) *If* $g(t) \in L(a, b), -\infty < a < b < \infty$, *then* $\gamma_\omega = \int_a^b g(t)\, e^{-i\omega t}\, dt \to 0$ *as* $\omega \to \infty$.

Transforming the variable of integration we may plainly suppose that $0 < a < b < 2\pi$. Putting $g(t) = 0$ outside (a, b), and applying the device of § 2.21, we obtain, for ω large enough, $2\,|\gamma_\omega| \leqslant \int_0^{2\pi} |g(t) - g(t + \pi/\omega)|\, dt \to 0$.

(ii) *If* $g(t) = f(t)\, h_x(t)$, *where* $f \in L(a, b)$, *and* $h_x(t)$, $a \leqslant t \leqslant b$, *is a uniformly bounded and uniformly continuous function depending on a parameter* x, *then* $\gamma_\omega \to 0$ *uniformly in* x.

Suppose that $0 < \varepsilon < a < b < 2\pi - \varepsilon$, and put $f(t) = 0$ outside (a, b). Let $h_x(t)$ be equal to 0 for $t \leqslant \varepsilon$ and $t \geqslant 2\pi - \varepsilon$, and be linear for $\varepsilon \leqslant t \leqslant a$, $b \leqslant t \leqslant 2\pi - \varepsilon$. The new function $h_x(t)$ is uniformly bounded and uniformly continuous and, since

$$\int_0^{2\pi} |f(t) - f(t + \pi/\omega)|\, dt \to 0, \qquad \operatorname*{Max}_{t,\,x} |h_x(t) - h_x(t + \pi/\omega)| \to 0,$$

the integral majorizing $2\,|\gamma_\omega|$ tends uniformly to 0 as $\omega \to \infty$. It is plain that *the result holds if* $h(t)$ *depends on more than one parameter.*

(iii) *Under the conditions of Theorem* 12.1, *the difference* $S_\omega(x) - S_{[\omega]}(x)$ *tends uniformly to* 0 *for* $x \in J_0'$. For, if $[\omega] = n$, $\omega - n = u$, then

$$(1) \qquad S_\omega(x) - S_n(x) = \Re\left\{ \frac{e^{i(n+\frac{1}{2}u)x}}{\pi} \int_{-\infty}^{\infty} f(t) \frac{2 \sin \frac{1}{2} u\,(x-t)}{x-t}\, e^{-i(n+\frac{1}{2}u)t}\, dt \right\}.$$

To show that the last integral tends uniformly to 0, we break it up into two integrals P and Q, where P is extended over some interval $(-A, A)$, and Q over $(-\infty, -A) + (A, \infty)$. If A is large enough, then $|Q| < \frac{1}{2}\varepsilon$ for $x \in J_0'$. Since the function $h_{x,u}(t) = 2 \sin \frac{1}{2} u(x-t)/(x-t)$ is uniformly continuous and uniformly bounded for $0 \leqslant u < 1$ and $x \in J_0'$, an application of (ii) shows that $|P| \to 0$, i. e. $|P + Q| < \varepsilon$ for $\omega > \omega_0$. This proves the lemma.

A moment's consideration shows that Theorem 12.1 is a consequence of (iii) and of the following lemma:

(iv) *Let* $f(x) = f'(x) + f''(x)$, *where* $f'(x) = f(x)$ *for* $x \in J_0$, $f'(x) = 0$ *for* $x \in \bar{J}_0$. *Then* $\delta_n = S_n(x; f') - s_n^*(x; f_0) \to 0$, $S_n(x; f'') \to 0$ *as* $n \to \infty$, *uniformly in* $x \in J_0'$.

Let $h_x(t)$ be a function of period 2π, equal to $1/(x-t) - \frac{1}{2} \operatorname{ctg} \frac{1}{2}(x-t)$ for $0 \leqslant t < 2\pi$. Since $\mathfrak{M}[h_x(t+\eta) - h_x(t); 0, 2\pi] \to 0$ with η, uniformly in $x \in J_0'$, an argument similar to that of § 2.501 shows that $\delta_n = \Im \frac{e^{inx}}{\pi} \int_0^{2\pi} f(t)\, h_x(t)\, e^{-int}\, dt \to 0$ uniformly in $x \in J_0'$. On the other hand, $S_n(x; f'') = U_n + V_n$, where U_n is equal to

$\Im \frac{e^{inx}}{\pi} \int\limits_{2\pi}^{\infty} f(t) \frac{e^{-int}}{x-t} dt$, and V_n is a similar integral formed for the interval $(-\infty, 0)$. To show that $|U_n| + |V_n| \to 0$, we proceed as in the proof of (iii).

This completes the proof of Theorem 12.1.

12.12. *Theorem* 12.1 *holds if $f(t)$ is integrable over any finite interval and if, moreover, $f(t)/t$ tends to* 0 *as $t \to \pm\infty$ and is of bounded variation in the neighbourhood of $t = \pm\infty$* [1]).

This last condition means that there is a number $B > 0$ such that f is of bounded variation in $(-\infty, -B)$ and in (B, ∞). Without loss of generality we may assume that $f(t)/t$ tends monotonically to 0 as $t \to \pm\infty$, $|t| > B$, for every f satisfying the conditions of the theorem is a sum of two functions satisfying this more stringent condition.

The proof of the theorem runs close to that of Theorem 12.1, and we need not repeat the whole argument. The proof of the latter theorem was based on Lemmas (iii) and (iv) of § 12.11. Those lemmas hold under new conditions, but in the proofs we must now apply the second mean-value theorem. For example, to prove Lemma 12.11(iii), we break up the right-hand side of 12.11(1) into three integrals extended over $(-\infty, -A)$, $(-A, A)$, and (A, ∞) respectively. The last of them is equal to the limit, for $A' \to \infty$, of the expression

$$(1) \qquad \frac{1}{\pi} \int\limits_{A}^{A'} \frac{f(t)}{t} \cdot \left\{1 + \frac{x}{t-x}\right\} [\sin \omega (t-x) - \sin n (t-x)]\, dt.$$

Applying the second mean-value theorem to the factors $f(t)/t$ and $1/(t-x)$, and observing that $f(t)/t \to 0$ as $t \to \infty$, we see that (1) tends to 0 as $A \to \infty$, $A' \to \infty$. This shows that the integral over (A, ∞) exists *and that it tends to* 0 *as $A \to \infty$, uniformly in $x \in J_0$ and $\omega \geqslant 1$.* The reader will have no difficulty in completing the proof.

12.2. Fourier's repeated integral. Suppose that $|f(t)|$ is integrable over $(-\infty, \infty)$. Then the right-hand side of 12.1(1) is equal to

$$(1) \quad \frac{1}{\pi} \int\limits_{-\infty}^{\infty} f(t)\, dt \int\limits_{0}^{\omega} \cos s(x-t)\, ds = \frac{1}{\pi} \int\limits_{0}^{\omega} ds \int\limits_{-\infty}^{\infty} f(t) \cos s(x-t)\, dt,$$

the inversion of the order of integration being clearly justified. Hence $S_\omega(x)$ is a partial integral of the infinite integral

---

[1]) Pringsheim [2]. The condition that $f(t)/t \to 0$ with $1/t$, is necessary, for, if e. g. $f(t) = t$, the integral 12.1(1) diverges.

$$(2) \qquad \frac{1}{\pi}\int_0^{\infty} ds \int_{-\infty}^{\infty} f(t)\cos s(x-t)\,dt =$$

$$(3) \qquad = \int_0^{\infty}(a_s\cos sx + b_s\sin sx)\,ds = \int_{-\infty}^{\infty} c_s e^{isx}\,ds,$$

where

$$(4) \qquad a_s = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos st\,dt, \qquad b_s = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin st\,dt,$$

$$(5) \qquad c_s = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\,e^{-ist}\,dt = \tfrac{1}{2}(a_s - ib_s).$$

The expressions a_s, b_s, c_s are analogous to Fourier coefficients; but s is a continuous variable and we obtain a trigonometrical integral of the form (3) instead of a trigonometrical series. Given a function $f(t)$, $-\infty < t < \infty$, such that the integrals (4) have a meaning, we may consider the integral (3) or, what is the same thing, the integral (2), and ask in what sense does it represent $f(x)$. The integral (2) is called *Fourier's repeated integral*. It is plain that if we have (1) for every ω, then the partial integrals $S_\omega(x)$ of (2) are given by the formula 12.1(1), i. e. the problem reduces to that of representing the function by means of Fourier's single integral, a problem which, in the most important cases, is settled by Theorems 12.1 and 12.2. The formula (1), however, is true only under certain conditions bearing on the behaviour of $f(t)$ not at individual points but in the whole interval $(-\infty, \infty)$; more precisely, in the neighbourhood of $t = \pm\infty$. This causes the range of application of Fourier's repeated integral to be more restricted than that of Fourier's single integral [1]). The formula (1) is certainly true when $|f| \in L(-\infty, \infty)$, and so, in view of Theorem 12.1, we have: *If* $|f| \in L(-\infty, \infty)$, *then* $S_\omega(x; f) - s_{[\omega]}(x; f_a) \to 0$, *uniformly in* $x \in J'_a$, *where* $S_\omega(x)$ *denotes the partial integral of* (2), *and* f_a, J_a, *and* J'_a *have the same meaning as before.*

---

[1]) The range of validity of Fourier's repeated integral can be considerably extended if we suppose that the integrals (4) are *summable* in some sense, e. g. summable (C, k) (§ 12.3). We shall not consider this problem here.

12.21. *The last theorem holds if $f(t)$ is integrable over any finite interval, tends to 0 with $1/t$, and is of bounded variation in the neighbourhood of $t = \pm\infty$.*

Assuming, as we may, that $f(t)$ tends monotonically to 0 as $t \to \pm\infty$, $|t| \geqslant B > 0$, by means of the second mean-value theorem we verify that the inner integral on the right of 12.2(1) converges for every $s > 0$ (but not necessarily for $s = 0$), and that the convergence is uniform over any range $0 < \delta \leqslant s \leqslant \omega$. Hence

$$(1)\quad \frac{1}{\pi}\int_{\delta}^{\omega} ds \int_{-\infty}^{\infty} f(t)\cos s(x-t)\,dt = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\frac{\sin\omega(x-t)}{x-t}\,dt - \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\frac{\sin\delta(x-t)}{x-t}\,dt.$$

We will show that the second integral on the right tends to 0 with δ. For the proof we break up the integral over $(-\infty, \infty)$ into three integrals, extended over $(-\infty, -A)$, $(-A, A)$, and (A, ∞) respectively. Since the integral of $(\sin u)/u$ over any finite interval is bounded, an application of the second mean-value theorem shows that, if $A > B$ is large enough, the first and the third of the three integrals are numerically less than a given $\varepsilon > 0$. Since, for fixed A, the second integral tends to 0 with δ, the last integral on the right of (1) is less than 3ε in absolute value for δ small enough, i. e. it tends to 0. *Thence we obtain* 12.2(1) *(and so also the theorem), where however the outer integral on the right is an improper integral:* $\int_{0}^{\omega} = \lim_{\delta\to+0}\int_{\delta}^{\omega}$.[1] That this is essential, and that $g(s) = \int_{-\infty}^{\infty} f(t)\cos s(x-t)\,dt$, considered as a function of s, may be non-integrable (in the Lebesgue sense), in the neighbourhood of $s = 0$, may be seen from the following example. There is a sequence $a_1 \geqslant a_2 \geqslant \ldots \to 0$ such that the sum of the series $\Sigma a_n \cos ns$ is not integrable in the neighbourhood of $s = 0$ (§ 5.121). Let $x = 0$, $f(t) = a_n$ for $n - \frac{1}{2} \leqslant t < n + \frac{1}{2}$, $n = 1, 2, \ldots$, $f(t) = 0$ for $0 \leqslant t < \frac{1}{2}$, $f(-t) = f(t)$. Then $sg(s)/4\sin\frac{1}{2}s = \Sigma a_n \cos ns$, and $g(s)$ is not integrable in the neighbourhood of $s = 0$ (see also § 5.7.4).

This result shows that, *under the hypotheses of the theorem stated at the beginning of the section, the outer integral in Fourier's repeated integral must be understood in the sense* $\lim\limits_{\substack{\omega\to\infty\\ \delta\to 0}}\int_{\delta}^{\omega}$.

12.3. Summability of integrals. So far we applied summability to series only, but a similar theory can be constructed for integrals. We start with the following lemma.

Let $\varphi(x)$ and $\psi(x)$ be two functions defined for $x \geqslant 0$ and integrable over any finite interval $(0, a)$; suppose that $\psi(x) > 0$ for $x > 0$ and let $\Phi(x)$ and $\Psi(x)$ denote respectively the integrals of $\varphi(t)$ and $\psi(t)$ over the interval $(0, x)$. Then, if $\Psi(x) \to \infty$ and $\varphi(x)/\psi(x) \to s$ as $x \to \infty$, we have $\Phi(x)/\Psi(x) \to s$.

For $s = 0$ the lemma was established in § 1.71. If we apply that result to the functions $\varphi_1(x) = \varphi(x) - s\psi(x)$ and $\psi_1(x) = \psi(x)$, we obtain the general result.

[1] Cf. Tonelli, *Serie trigonometriche*, p. 413.

We write $\Phi_0(x) = \varphi(x)$, and denote by $\Phi_k(x)$, $k = 1, 2, \ldots$, the integral of $\Phi_{k-1}(t)$ over $0 \leqslant t \leqslant x$. Similarly we define $\Psi_k(x)$. It is plain that, if $\Phi_k(x)/\Psi_k(x) \to s$ as $x \to \infty$, then $\Phi_l(x)/\Psi_l(x) \to s$ for $l \geqslant k$. Suppose that $\varphi(x) = 1$; then $\Psi_k(x) = x^k/k!$ We shall say that s is the (C, k) limit of $\varphi(x)$ as $x \to \infty$ and write $(C, k)\,\varphi(x) \to s$, if $\Phi_k(x)\,k!/x^k \to s$, i. e. if

$$(1) \qquad k x^{-k} \int_0^x (x-t)^{k-1} \varphi(t)\, dt \to s \quad \text{as} \quad x \to \infty.$$

Now we may take (1) as the definition of the (C, k) limit for every $k > 0$, integral or fractional. By the $(C, 0)$ limit of the function $\varphi(x)$ as $x \to \infty$, we mean the ordinary limit. Since φ is integrable over any finite interval $(0, a)$, the left-hand side of (1) exists for almost every x (this follows from the results of § 2.11) and is itself integrable over $(0, a)$. If $\varphi(t)$ is bounded over any finite interval—a most frequent case in applications—the left-hand side of (1) exists everywhere.

(i) *If $\alpha \geqslant 0$, $\beta > 0$, and if $(C, \alpha)\,\varphi(x) \to s$, then $(C, \alpha+\beta)\,\varphi(x) \to s$.*

We assume that $\alpha > 0$. In the argument we shall require the formula

$$(2) \qquad \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\, du = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

a proof of which will be found in most text-books of Analysis [1]). Let us denote the left-hand side of (1) by $k\Phi_k^*(x)/x^k$. We begin by proving that

$$(3) \qquad \Phi^*_{\alpha+\beta}(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x \Phi^*_\alpha(t)\,(x-t)^{\beta-1}\, dt.$$

For the integral on the right of (3) is equal to

$$\int_0^x (x-t)^{\beta-1} dt \int_0^t \varphi(u)\,(t-u)^{\alpha-1}\, du = \int_0^x \varphi(u)\, du \left\{ \int_u^x (x-t)^{\beta-1} (t-u)^{\alpha-1}\, dt \right\}.$$

Thence, transforming the variables in the inner integral on the right, and using (2), we obtain (3).

Now, if $(C, \alpha)\,\varphi(x) \to s$, then $\Phi^*_\alpha(t) = st^\alpha/\alpha + \varepsilon(t)\,t^\alpha$, where $\varepsilon(t) \to 0$ as $t \to \infty$. Since $\Gamma(\lambda+1) = \lambda\Gamma(\lambda)$, we obtain from (3)

$$(4) \qquad (\alpha+\beta)\,\Phi^*_{\alpha+\beta}(x)/x^{\alpha+\beta} = s + Cx^{-\alpha-\beta} \int_0^x \varepsilon(t)\, t^\alpha (x-t)^{\beta-1}\, dt,$$

where C denotes a constant. Let $\varepsilon > 0$ be an arbitrarily small number and let $|\varepsilon(t)| < \varepsilon$ for $t > x_0$. Breaking up the last integral into two, extended

---

[1]) The formula (2) can also be obtained from 3.11(3) and the relation $A_n^\alpha \simeq n^\alpha/\Gamma(\alpha+1)$.

over $(0, x_0)$ and (x_0, x) respectively, the reader will have no difficulty in proving that the right-hand side of (4) tends to s. This completes the proof of the theorem for $\alpha > 0$ [1]. The case $\alpha = 0$ is still simpler.

In the foregoing discussion we supposed that $x \to \infty$, but a similar theory may be developed in the case of x tending to any other limit. For example, the (C, k) limit of $\varphi(x)$ for $x \to +0$ may also be defined by (1), with the difference that in that formula x now tends to $+0$.

Given an integral $J = \int_0^\infty f(t)\,dt$, we shall say that it is summable (C, k), to the value s, if we have (1) with $\varphi(x) = \int_0^x f(t)\,dt$, i. e. if

$$(5) \qquad x^{-k}\int_0^x (x-t)^k f(t)\,dt \to s \qquad \text{as } x \to \infty.$$

This definition presupposes that $f(x)$ is integrable over any finite interval. The left-hand side of (5) exists then for almost every x, even if $k > -1$. In view of (i), *summability (C, α) implies summability $(C, \alpha+\beta)$, $\alpha > 0$, $\beta > 0$, to the same value* [2].

Given an arbitrary series $(U)\, u_0 + u_1 + \dots$, let $U_n = u_0 + u_1 + \dots + u_n$, and let $U(x) = U_n$ for $n \leqslant x < n+1$, $n = 0, 1, \dots$ If, for $x \to \infty$, the (C, α) limit of $U(x)$ exists and is equal to s, we shall say that the series U is summable by M. Riesz's method of order α, or summable (R, α), to sum s. M. Riesz has shown that *the methods (R, α) and (C, α) are equivalent* [3], *i. e., if a series is summable by one of these methods, it is summable by the other to the same sum.* The proof of the general result is rather complicated, but the special case $\alpha = 1$, which is of independent importance, is fairly easy and may be left to the reader as an exercise.

Since, under the hypothesis of Theorem 12.1, the $(C, 1)$ limit of the difference $S_\omega(x) - s_{[\omega]}(x)$ exists and is equal to 0, and since Fourier series are summable $(C, 1)$ almost everywhere, we obtain:

Under the hypothesis of Theorem 12.1, the $(C, 1)$ limit of $S_\omega(x)$ exists almost everywhere and is equal to $f(x)$. In particular, this limit exists and is equal to $\frac{1}{2}[f(x+0) + f(x-0)]$ at every point of simple discontinuity of f. It exists uniformly over any finite interval at all points of which f is continuous.

In the same way we may complete Theorems 12.12 and 12.2. If we assume M. Riesz's equivalence theorem in its general form, we may replace summability $(C, 1)$ by (C, δ), $\delta > 0$. All these results can however be obtained independently of M. Riesz's theorem, by an argument similar to that of § 3.3 [4].

---

[1]) The result holds for $s = \infty$.

[2]) The result holds for $\alpha > -1$.

[3]) A proof will be found in Hobson's *Theory of Functions*, 2, p. 90.

[4]) See e. g. Hobson, *loc. cit.* p. 737.

12.4. Fourier transforms[1]. Changing the definition 12.2(5) slightly, we shall write

$$(1)\qquad F(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ixy}\, dx,$$

where $f(x)$ is now a complex function. When $f(x)$ is represented by Fourier's repeated integral 12.2(2), we have

$$(2)\qquad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(y)\, e^{ixy}\, dy,$$

the integral on the right being defined as $\lim\limits_{\omega\to\infty} \int\limits_{-\omega}^{\omega}$. The function $F(y)$ is called the Fourier *transform* of $f(x)$. It exists for every x if $f \in L(-\infty, \infty)$. We shall now prove that

(i) *If* $f(x) \in L^2(-\infty, \infty)$, *the integral in* (1) *converges, in a certain sense, to a function* $F(y) \in L^2(-\infty, \infty)$. *The function* F *satisfies* (2) *and the relation*

$$(3)\qquad \int_{-\infty}^{\infty} |F(y)|^2\, dy = \int_{-\infty}^{\infty} |f(x)|^2\, dx.$$

Let S denote the set of step-functions $f(x)$ which vanish for $|x|$ large. If $f \in S$, we define F by the formula (1); in all other cases we shall define F indirectly. We begin by proving (3) for $f \in S$. Then, for $\omega > 0$,

$$(4)\qquad \int_{-\omega}^{\omega} |F(y)|^2\, dy = \frac{1}{2\pi} \int_{-\omega}^{\omega} dy \int_{-\infty}^{\infty} f(x)\, e^{-ixy}\, dx \int_{-\infty}^{\infty} \bar{f}(x')\, e^{ix'y}\, dx' =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)\, \bar{f}(x')\, dx\, dx' \int_{-\omega}^{\omega} e^{iy(x'-x)}\, dy =$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)\, \bar{f}(x')\, \frac{\sin \omega (x - x')}{x - x'}\, dx\, dx' = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x)\, S_\omega(x; \bar{f})\, dx,$$

where S_ω is defined by 12.1(1). The above transformations are perfectly legitimate since the integrals are infinite in appearance

[1]) The results of this section are due to Plancherel [1], [2]; see also F. Riesz [9]. Interesting generalizations will be found in Watson [1].

only. Now observe that, in the case considered, $S_\omega(x; \bar{f})$ is uniformly bounded in x and ω, and tends to $\bar{f}(x)$ as $\omega \to \infty$, except, perhaps, at a finite number of points. It is sufficient to consider the case when f is the characteristic function of an interval. But then the result follows (independently of the more difficult Theorem 12.1) from the formula $\int_0^\infty \frac{\sin t}{t}\, dt = \frac{1}{2}\pi$ and from the fact that the partial integrals of the last integral are bounded. Comparing the extreme terms of (4), and making $\omega \to \infty$, we obtain the equation (3), by Lebesgue's theorem on the integration of bounded sequences.

The formula (1) defines an additive operation $F = T[f]$. This operation is actually defined for functions f belonging to a set S, which is everywhere dense in the space $L^2(-\infty, \infty)$ [1]. Hence, in view of the formula (3), valid for $f \in S$, and the remarks of § 9.22 [2], the operation $T[f]$ may be extended, by continuity, to the whole space $L^2(-\infty, \infty)$, and this extension is unique. This operation is of type (2,2) and its modulus is equal to 1. This means that, for every $f \in L^2(-\infty, \infty)$, we have (3) with '=' replaced by '$\leqslant$'. To prove that equality actually occurs, let $f \in L^2$, $f_n \in S$, $n = 1, 2, \ldots$, $\mathfrak{M}_2[f - f_n] \to 0$ [3], $F_n = T[f_n]$. Since $\mathfrak{M}_2[F - F_n] \leqslant \mathfrak{M}_2[f - f_n] \to 0$, Minkowski's inequality gives: $\mathfrak{M}_2[f_n] \to \mathfrak{M}_2[f]$, $\mathfrak{M}_2[F_n] \to \mathfrak{M}_2[F]$. This and the equations $\mathfrak{M}_2[f_n] = \mathfrak{M}_2[F_n]$ imply $\mathfrak{M}_2[f] = \mathfrak{M}_2[F]$, i. e. (3).

It remains to prove (2), which may be writen $f(x) = T^*T[f]$, where T^* denotes the operation we obtain from T by changing the sign of y. Since the operations T and T^* are continuous in the space $L^2(-\infty, \infty)$, it is sufficient to prove the relation $f = T^*T[f]$ when $f \in S$, or, still simpler, when f is the characteristic function of an interval (a, b). Then $F(y) = i(e^{-iyb} - e^{-iya})/\sqrt{2\pi}\, y$, and

---

[1]) This is a special case of the more general and difficult Theorem 9.21(i). An independent proof runs as follows. Let S_1 be the set of functions $f(x) \in L^2(-\infty, \infty)$ which vanish for $|x|$ large. S_1 is dense in $L^2(-\infty, \infty)$, and so it is enough to show that S is dense in S_1. Let $f \in S_1$; transforming the variable x, we may suppose that $f(x)$ vanishes outside $(0, 2\pi)$. Then there is a continuous function $\sigma(x)$ such that $\mathfrak{M}_2[f - \sigma; 0, 2\pi] < \frac{1}{2}\varepsilon$ (§ 4.21(1)). If $s(x)$ is a step-function vanishing outside $(0, 2\pi)$, and such that $\mathfrak{M}_2[\sigma - s; 0, 2\pi] < \frac{1}{2}\varepsilon$, then $\mathfrak{M}_2[f - s; -\infty, \infty] < \varepsilon$.

[2]) The Stieltjes-Lebesgue integrals considered there reduce to ordinary Lebesgue integrals.

[3]) We write $\mathfrak{M}_2[g]$ instead of $\mathfrak{M}_2[g; -\infty, \infty]$.

$$(5)\qquad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(y)\, e^{ixy}\, dy = \frac{1}{\pi}\int_0^{\infty} \frac{\sin(x-a)y}{y}\, dy + \frac{1}{\pi}\int_0^{\infty} \frac{\sin(b-x)y}{y}\, dy,$$

i. e. the left-hand side of (5) is equal to 1 for $a < x < b$, and to 0 for $x < a$ and $x > b$. This completes the proof of (i), if we take for granted the result, which will be established below, that, whenever the integral defining the transform of a function $g \in L^2$ converges almost everywhere, it converges to $T[g]$.

In the previous argument, the operation $T[f]$ was defined directly by (1) only when $f \in S$; to define $T[f]$ for general f we used continuity. We shall now show that, if $g \in L^2(-\infty, \infty)$ vanishes outside some interval $(-A, A)$, then $T[g]$ may still be defined by the formula (1). For let $G(y) = T[g]$ and let $G^*(y)$ be the value of the integral in (1), with f replaced by g. Let $g_n(x)$, $n = 1, 2, \ldots$ be a sequence of step-functions vanishing outside $(-A, A)$ and such that $\mathfrak{M}_2[g - g_n] \to 0$. If $G_n(y) = T[g_n]$, then $\mathfrak{M}_2[G - G_n] \to 0$ and, à fortiori, $\mathfrak{M}_2[G - G_n;\, -\omega, \omega] \to 0$, for every $\omega > 0$. On the other hand, Schwarz's inequality shows that $G_n(y)$ tends uniformly to $G^*(y)$ over any interval, and so $\mathfrak{M}_2[G^* - G_n;\, -\omega, \omega] \to 0$. This and the relation $\mathfrak{M}_2[G - G_n;\, -\omega, \omega] \to 0$ show that $G^*(y) = G(y)$ for $-\omega < y < \omega$, and so also for $-\infty < y < \infty$.

Let $f \in L^2(-\infty, \infty)$ and $\omega > 0$. We write

$$(6)\qquad F_\omega(y) = \frac{1}{\sqrt{2\pi}}\int_{-\omega}^{\omega} f(x)\, e^{-ixy}\, dx;$$

then $F_\omega(y) = T[f_\omega]$, where $f_\omega(x)$ is equal to $f(x)$ for $|x| < \omega$, and to 0 elsewhere. Since $\mathfrak{M}_2[F_\omega - T[f]] = \mathfrak{M}_2[f_\omega - f] \to 0$ as $\omega \to \infty$, proposition (i) may be restated as follows:

(ii) *For every $f \in L^2(-\infty, \infty)$, the integral in* (1) *converges in mean to a function $F(y) \in L^2(-\infty, \infty)$, that is $\mathfrak{M}_2[F - F_\omega] \to 0$ as $\omega \to \infty$. The integral in* (2) *converges in mean to $f(x)$, and F and f satisfy the Parseval relation* (3).

Since $\mathfrak{M}_2[F - F_\omega] \to 0$, there exists a sequence $\{\omega_k\}$ such that $F_{\omega_k}(y) \to F(y)$ for almost every y (§ 4.2). Therefore, *if the integral in* (1) *converges almost everywhere, it converges to the transform of f.*

It is not difficult to obtain a formula for $F(y)$. Let $\Phi(y)$ and $\Phi_\omega(y)$ denote the integrals of F and F_ω over $(0, y)$. By Schwarz's inequality, $|\Phi(y) - \Phi_\omega(y)| \leqslant y^{1/2}\, \mathfrak{M}_2[F - F_\omega;\, 0, y] \to 0$,

i. e. $\Phi(y) = \lim_{\omega\to\infty} \Phi_\omega(y)$. Since $\Phi_\omega(y) = \frac{1}{\sqrt{2\pi}} \int_{-\omega}^{\omega} f(x) \frac{e^{-ixy}-1}{-ix}\, dx$, and $\Phi'(y) = F(y)$ for almost every y, we obtain the first of the formulae

$$(7)\quad F(y) = \frac{d}{dy}\left\{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \frac{e^{-ixy}-1}{-ix}\, dx\right\},\quad f(x) = \frac{d}{dx}\left\{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(y) \frac{e^{ixy}-1}{iy}\, dy\right\}.$$

The second formula, which corresponds to (2), may be obtained similarly.

The formula (2) tells us that to every $f(x) \in L^2(-\infty, \infty)$ corresponds a function $g(y) \in L^2(-\infty, \infty)$, whose transform is $f(x)$ (an analogue of the Riesz-Fischer theorem). It suffices to put $g(y) = F(-y)$, where $F(y)$ is the transform of $f(x)$.

12.41. *If* $f(x) \in L^p(-\infty, \infty)$, $1 < p \leqslant 2$, *the integral in* 12.4(1) *converges in mean, with index* $p' = p/(p-1)$, *to a function* $F(y) \in L^{p'}$, *which satisfies the equations* 12.4(7) *and the inequality*

$$(1)\qquad \left\{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |F(y)|^{p'}\, dy\right\}^{1/p'} \leqslant \left\{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)|^{p}\, dx\right\}^{1/p}.^{1)}$$

This is an extension to Fourier integrals of Theorem 9.1(a). We first observe that the formula 12.4(1) defines a functional operation $F = T[f]$, when $f \in L(-\infty, \infty)$ or $f \in L^2(-\infty, \infty)$. Using the notation of § 9.22, we may say that T is of type $(1, \infty)$ and of type $(2, 2)$, and that $M_{1,0} = (2\pi)^{-1/2}$, $M_{1/2, 1/2} = 1$. Hence, by Theorem 9.23, the operation may be extended, so as to become of type (p, p'); and $M_{1/p, 1/p'} \leqslant (2\pi)^{1/2 - 1/p}$. This gives (1), where $F = T[f]$.

Let f_ω have the same meaning as in § 12.4. If $f \in L^p$, then $f_\omega \in L$, and so $F_\omega = T[f_\omega]$ is given by the formula 12.4(6). Since $\mathfrak{M}_{p'}[T[f] - F_\omega] \leqslant M_{1/p, 1/p'}\, \mathfrak{M}_p[f - f_\omega] \to 0$, the integral in 12.4(1) converges in mean, with index p', to a function $F(y) \in L^{p'}$. Arguing as in § 12.4, and using Hölder's inequality instead of Schwarz's, we obtain the first formula 12.4(7) (cf. also § 12.5.3).

To prove the second formula 12.4(7), observe that, if $f(x)$ is absolutely integrable over $(-\infty, \infty)$, then the Fourier integral of f may be integrated formally over any finite interval. This follows e. g. from the fact that Fourier series may be integrated

---

1) Titchmarsh [6]; see also M. Riesz [3].

formally and Theorem 12.1. Since $f \in L^p(-\infty, \infty)$, $p > 1$, the function equal to $f(x)$ for $|x| < a$ and to 0 elsewhere belongs to $L(-\infty, \infty)$, and so, if $|x| < a$,

$$\int_0^x f(u)\,du = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ixy}-1}{iy}\left\{\int_{-a}^{a} f(t)\,e^{-iyt}\,dt\right\}dy = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{ixy}-1}{iy}F_a(y)\,dy.$$

Since $\mathfrak{M}_{p'}[F_a - F] \to 0$ as $a \to \infty$, an application of Hölder's inequality shows that we may replace $F_a(y)$ by $F(y)$ in the last integral, and the second formula 12.4(7) follows. This completes the proof of the theorem.

12.42. The result which we obtained is, in one respect, incomplete. Whereas it was proved that the integral in 12.4(1) converges in mean, with index p', the reciprocal relation 12.4(2) was established only in the sense of the second formula 12.4(7). This result was completed by Hille and Tamarkin [3], who showed that *the integral* 12.4(2) *converges in mean, with index p, to f(x).* This theorem is suggested by Theorem 7.3(i), if we observe that the function $F(y)$ is an analogue of a sequence of Fourier coefficients, and the partial integrals of the integral 12.4(2) play the rôle of the partial sums of a Fourier series. The proof is based on the following lemma:

If $f \in L^r(-\infty, \infty)$, $r > 1$, the function

$$(1)\quad g(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{x-t}\,dt = -\lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{\varepsilon}^{\infty}\frac{f(x+t)-f(x-t)}{t}\,dt$$

exists for almost every x and satisfies an inequality $\mathfrak{M}_r[g] \leqslant A_r \mathfrak{M}_r[f]$, where A_r depends on r only[1]).

Since, in view of Hölder's inequality, the function $f(t)/(t-x)$ is integrable in the neighbourhood of $t = \pm\infty$, the first part of the lemma follows from Theorem 7.1(i). To prove the second part, we put $g_n(x) = \dfrac{1}{2\pi n}\displaystyle\int_{-\pi n}^{\pi n} f(t)\,\mathrm{ctg}\,\frac{x-t}{2n}\,dt$ and consider the difference $\delta_n = g(x) - g_n(x)$. Then

$$\delta_n = \frac{1}{2\pi n}\int_{-\pi n}^{\pi n} f(t)\left[\frac{2n}{x-t} - \mathrm{ctg}\,\frac{x-t}{2n}\right]dt +$$

---

[1]) M. Riesz [4].

$$+\frac{1}{\pi}\int_{-\infty}^{-\pi n}\frac{f(t)}{x-t}\,dt+\frac{1}{\pi}\int_{\pi n}^{\infty}\frac{f(t)}{x-t}\,dt=\alpha_n+\beta_n+\gamma_n.$$

The expressions α_n, β_n, and γ_n tend to 0 as $n\to\infty$ [1]); hence $\delta_n\to 0$, $g_n(x)\to g(x)$, and an application of Fatou's lemma to the inequality $\mathfrak{M}_r[g_n;-\pi n,\pi n]\leqslant A_r\mathfrak{M}_r[f;-\pi n,\pi n]\leqslant A_r\mathfrak{M}_r[f]$ (§ 7.21) shows that $\mathfrak{M}_r[g;-w,w]\leqslant A_r\mathfrak{M}_r[f]$ for every $w>0$, i. e. $\mathfrak{M}_r[g]\leqslant A_r\mathfrak{M}_r[f]$. This completes the proof of the lemma.

If $F_\omega(y)$ is given by 12.4(6), and $w>0$ is any finite number, then

$$\frac{1}{\sqrt{2\pi}}\int_{-w}^{w}F_\omega(y)\,e^{ixy}\,dy=\frac{1}{\pi}\int_{-\omega}^{\omega}f(t)\,\frac{\sin w\,(x-t)}{x-t}\,dt.$$

Since $F_\omega(y)$ tends in mean, with index p', to $F(y)$, we may put $\omega=\infty$ in the last equation, and we obtain

$$(2)\qquad \frac{1}{\sqrt{2\pi}}\int_{-w}^{w}F(y)\,e^{ixy}\,dy=\frac{1}{\pi}\int_{-\infty}^{\infty}f(t)\,\frac{\sin w\,(x-t)}{x-t}\,dt.$$

Applying the lemma and using the same device as in § 7.3, we obtain that the left-hand side $\Phi_w(x)$ of (2), satisfies the inequality $\mathfrak{M}_p[\Phi_w]\leqslant 2A_p\mathfrak{M}_p[f]$. To show that $\mathfrak{M}_p[\Phi_w-f]\to 0$ as $w\to\infty$, we put $f=f'+f''$, and, correspondingly, $\Phi_w=\Phi'_w+\Phi''_w$, where $f'\in S$ (§ 12.4) and $\mathfrak{M}_p[f'']<\varepsilon$; then

$$\mathfrak{M}_p[f-\Phi_w]\leqslant\mathfrak{M}_p[f'-\Phi'_w]+\mathfrak{M}_p[f'']+\mathfrak{M}_p[\Phi''_w]\leqslant\mathfrak{M}_p[f'-\Phi'_w]+(2A_p+1)\,\varepsilon,$$

and it is sufficient to show that $\mathfrak{M}_p[f'-\Phi'_w]\to 0$. We may restrict ourselves to the case when the function f', which we shall now denote by f again, is the characteristic function of an interval (a,b). Then $F(y)=i\,(e^{-iyb}-e^{-iya})/\sqrt{2\pi}\,y$, and the second mean-value theorem shows that

$$\Phi_w(x)=\frac{1}{\pi}\left[\int_0^{(x-a)w}\frac{\sin y}{y}\,dy+\int_0^{(b-x)w}\frac{\sin y}{y}\,dy\right]=w^{-1}O\!\left(\frac{1}{|x|}\right)$$

for $|x|$ large. Since $\mathfrak{M}_p[\Phi_w-f;-A,A]$ tends to 0 for any fixed A, and $\mathfrak{M}_p[\Phi_w-f;-\infty,-A]+\mathfrak{M}_p[\Phi_w-f;A,\infty]$ is small for A large, it is easy to see that $\mathfrak{M}_p[\Phi_w-f]\to 0$, and the theorem is established [2]).

---

[1]) Since $|1/u-\operatorname{ctg}u|\leqslant C<\infty$ for $|u|<{}^3/_4\,\pi$, we obtain that, for fixed x, and n large enough, $|\alpha_n|\leqslant C\,\mathfrak{M}[f;-\pi n,\pi n]/2\pi n\leqslant C\,\mathfrak{M}_r[f](2\pi n)^{-1/r}\to 0$.

[2]) Hence any function $f\in L^p(-\infty,\infty)$, $1<p\leqslant 2$, is the transform of a function $g\in L^{p'}(-\infty,\infty)$.

10.5. Miscellaneous theorems and examples.

1. If $f(x) \in L^2(-\infty, \infty)$, the integral $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iyx}\, dx$ is summable $(C, 1)$ for almost every y. Plancherel [2].

[Observe that $f(x)$ is the transform of a function of the class $L^2(-\infty, \infty)$].

2. If $f(x) \in L^2(-\infty, \infty)$, then $\frac{1}{\sqrt{2\pi}} \int_{-\omega}^{\omega} f(x) e^{-iyx} dx = o(\sqrt{\log \omega})$, for almost every y.

[Use the method of § 10.32].

3. If $f(x) \in L^q(-\infty, \infty)$, $q > 2$, the function

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \frac{e^{-ixy} - 1}{-ix}\, dx$$

may be almost everywhere non-differentiable.

[Let $\{\alpha_n\}$ be a sequence of real numbers such that $\Sigma |\alpha_n|^q < \infty$, $\Sigma \alpha_n^2 = \infty$; put $f(x) = \alpha_n$ for $2^n - \frac{1}{2} < |x| < 2^n + \frac{1}{2}$, $n = 0, 1, \dots$, and $f(x) = 0$ elsewhere, and apply Theorem 5.7.7. For a similar result see Titchmarsh [6]].

4. Show that Mellin's inversion formulae

$$\varphi(s) = \int_0^{\infty} x^{s-1} \psi(x)\, dx, \quad \psi(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \varphi(s) x^{-s}\, ds,$$

may, with suitable conditions, be deduced from the formulae 12.4(1) and 12.4(2).

TERMINOLOGICAL INDEX

(Numbers refer to pages).

NOTATIONS.

BIBLIOGRAPHY.

Banach, S. *Opérations linéaires*, Monografje Matematyczne, I, Warszawa, 1932.

Carslaw, H. S. *Introduction to the Theory of Fourier's Series and Integrals*, London, ed. 3, 1930.

Evans, G. *The Logarithmic Potential, Discontinuous Dirichlet and Neumann Problems*, New York, 1927.

Hardy, G. H., Littlewood, J. E., and Pólya, G., *Inequalities*, Cambridge, 1934.

Hardy, G. H. and Riesz, M. *The General Theory of Dirichlet's series*, Cambridge, 1915.

Hausdorff, F. *Mengenlehre*, Berlin und Leipzig, ed. 2, 1927.

Hobson, E. W. *The Theory of Functions of a Real Variable*, Cambridge, Vol. 1 (ed. 3), 1927, Vol. 2 (ed. 2), 1926.

Landau, E. *Darstellung und Begründung einiger neueren Ergebnisse der Funktionentheorie*, ed. 2, Berlin, 1929.

Lebesgue, H. *Leçons sur l'intégration et sur la recherche des fonctions primitives*, Paris, ed. 2, 1928. — *Leçons sur les séries trigonométriques*, Paris, 1906.

Plessner, A. *Trigonometrische Reihen*, in Pascal's „Repertorium der höheren Analysis", Vol. I_3, Leipzig und Berlin, 1929.

Saks, S. *Théorie de l'intégrale*, Monografje Matematyczne, II, Warszawa, 1933.

Tamarkin, J. D. *Fourier Series*, Providence, 1933 (hektographed).

Titchmarsh, E. C. *The Theory of Functions*, Oxford, 1932.

Tonelli, L. *Serie Trigonometriche*, Bologna, 1928.

de la Vallée Poussin, C. *Intégrales de Lebesgue, fonctions d'ensemble, classes de Baire*, Paris, 1916. — *Cours d'analyse infinitésimale*, Louvain-Paris, ed. 5, 1923.

ORIGINAL MEMOIRS REFERRED TO IN THE TEXT.

ABBREVIATIONS: A. E. N. = Annales de l'Ecole Normale Supérieure. **A. f. M.** = Arkiv för Matematik, Astronomi och Fysik. — **A. M.** = Acta Mathematica. — **Ann. M.** = Annals of Mathematics. — **A. S.** = Acta Litterarum ac Scientiarum Regiae Universitatis Hungaricae Francisco-Josephinae, Szeged. — **B. A. B.** = Sitzungsberichte der Preussischen Akademie der Wissenschaften zu Berlin. — **B. A. M. S.** = Bulletin of the American Math. Society. — **B. A. P.** = Bulletin International de l'Académie Polonaise, classe A de sciences mathématiques et naturelles, Cracovie. — **B. S. F. M.** = Bulletin de la Société Math. de France. — **C. R.** = Comptes Rendus de l'Académie de Sciences de Paris. — **F. M.** = Fundamenta Mathematicae. — **G. N.** = Göttinger Nachrichten. — **J. D. M. V.** = Jahresbericht der Deutschen Mathematiker Vereinigung. — **J. L. M. S.** = Journal of the London Math. Soc. — **J. M.** = Journal de Mathématiques. — **J. f. M.** = Journal für reine und angewandte Mathematik. — **M. A.** = Mathematische Annalen. — **M. M.** = Messenger of Mathematics. — **M. Z.** = Mathematische Zeitschrift. — **P. L. M. S.** = Proceedings of the London Math. Society. — **Q. J.** = Quarterly Journal of Mathematics. — **R. L.** = Rendiconti della Reale Accademia dei Lincei, Roma. — **R. P.** = Rendiconti del Circolo Matematico di Palermo. — **S. M.** = Studia Mathematica. — **T. A. M. S.** = Transactions of the American Math. Society. — **W. B.** = Sitzungsberichte der Math.-Naturwiss. Klasse der Akad. der Wiss. zu Wien.

Alexits, G. [1] Über den Grad der Annäherung durch die Cesàrosche Mittel der Fourierreihe, A. S., 3 (1927), 32—37.

Andersen, A. F. [1] Studier over Cesàros Summabilitetsmetode, Copenhague, 1921, p. 1—100.

Banach, S. [1] Über einige Eigenschaften der lakunären trigonometrischen Reihen, S. M., 2 (1930), 207—220.

Bary, N. [1] Sur l'unicité du développement trigonométrique, F. M., 9 (1927), 62—118.

Bernstein, S. [1] Sur l'ordre de la meilleure approximation des fonctions continues par des polynômes de degré donné, Mém. de l'Acad. Royale de Belgique, 2me serie, 4 (1912), 1—104. — [2] Sur la convergence absolue des séries trigonometriques, C. R., 158 (1914), 1661 — 1664 — [3] Sur la convergence absolue des séries trigonométriques. Communications de la Soc. Math. de Kharkow, 2me série, 14 (1914), 139—144 (in Russian).

Birnbaum, Z. and **Orlicz, W.** [1] Über die Verallgemeinerung des Begriffes der zueinander konjugierten Potenzen, S. M., 3 (1931), 1—67.

Bochner, S. [1] Über Faktorenfolgen für Fouriersche Reihen, A. S., 4 (1929) 125—129.

Boks, T. J. [1] Sur le rapport entre les méthodes d'intégration de Riemann et de Lebesgue, R. P., 45 (1921), 211—264.

Borgen, S. [1] Über $(C, 1)$ Summierbarkeit von Reihen orthogonaler Funktionen, M. A., 98 (1928), 125—150.

Bosanquet, L. S. [1] On the summability of Fourier series, P. L. M. S., 31 (1930), 144—164.

Cantor, G. [1] Über die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen, M. A., 5 (1872), 123—132.

Carathéodory, C. [1] Über den Variabilitätsbereich der Koeffizienten von Potenzreihen die gegebene Werte nicht annehmen, M. A., 64 (1907), 95—115. — [2] Über die Fourierschen Koeffizienten monotoner Funktionen, B. A. B., 30 (1920), 559—573.

Carleman, T. [1] Über die Fourierkoeffizienten einer stetigen Funktion, A. M., 41 (1918), 377—384. — [2] A theorem concerning Fourier series, P. L. M. S, 21 (1923), 483 — 492. — [3] Sur les équations intégrales singulières à noyau réel et symétrique, Uppsala Universitets Arsskrift, 1923, p. 1—228.

Chapman, S. [1] On non integral orders of summability of series and integrals, P. L. M. S., 9 (1911), 369—409.

Chaundy, T. W. and **Jollife, A. E.** [1] The uniform convergence of a certain class of trigonometrical series, P. L. M. S., 15 (1916), 214—216.

van der Corput, J. G. [1] Zahlentheoretische Abschätzungen, M. A., 81 (1921), 53—79.

Cramér, H. [1] Etudes sur la sommation des séries de Fourier, A. f. M., 13 (1919), No 20, 1—21.

Denjoy, A. [1] Sur l'absolue convergence des séries trigonométriques, C. R., 155 (1912), 135—136. — [2] Sur les ensembles parfaits présentant le caractère (A), A. L., 29, (1920), 316 — 318. — [3] Sur l'intégration riemannienne, C. R., 169 (1919), 219—221. — [4] Calcul des coefficients d'une série trigonométrique convergente quelconque dont la somme est donnée, C. R., 172 (1921), 1218—1221.

Du Bois-Reymond, P., [1] Untersuchungen über die Convergenz und Divergenz der Fourierschen Darstellungsformeln, Abhand. Akad. München, XII (1876), 1 — 103. — [2] Über die sprungweise Werthänderungen analytischer Funktionen, M. A., 7 (1874), 241—261. — [3] Beweis dass die Koeffizienten der trigonometrischen Reihe..., Abh. Akad. München, XII (1876), 117—166.

Faber, G. [1] Über stetige Funktionen, II, M. A., 69 (1910), 372—443.

Fatou, P. [1] Séries trigonométriques et séries de Taylor, A. M., 30 (1906), 335—400. — [2] Sur la convergence absolue des séries trigonométriques, B. S. M. F., 41 (1913), 47—53.

Fejér, L. [1] Untersuchungen über Fouriersche Reihen, M. A., 58 (1904), 501—569.— [2] Über die Positivität von Summen die nach trigonometrischen oder Legendreschen Funktionen fortschreiten, A. S., 2 (1925), 75—86. — [3] Über die Bestimmung des Sprunges der Funktion aus ihrer Fourierreihe, J. f. M., 142 (1913), 165—168. — [4] Neue Eigenschaften der Mittelwerte bei den Fourierreihen, J. L. M. S., 8 (1933), 53—62. — [5] Über gewisse durch die Fouriersche und Laplacesche Reihe definierten Mittelkurven und Mittelflächen, R. P., 38 (1914), 79—97.—[6] Über konjugierte trigonometrische Reihen, J. f. M., 144 (1913), 48—56. — [7] Sur les singularités des séries de Fourier de fonctions continues, A. E. S., 28 (1911), 63—103. — [8] Lebesguesche Konstanten und divergente Fourierreihen, J. f. M., 138 (1910), 22—53. — [9] Über gewisse Minimumprobleme der Funktionentheorie, M. A, 97 (1926), 104—123. — [10] Über die arithmetischen Mittel erster Ordnung der Fourierreihen, G. N., 1925, 13—17.

Fekete, M. [1] Über Faktorenfolgen welche die „Klasse" einer Fourierschen Reihe unverändert lassen, A. S., 1 (1923), 148—166.

Fichtenholz, G. [1] Sur l'intégrale de Poisson et quelques questions qui s'y rattachent, F. M., 13 (1929), 1—33.

Fischer, E. [1] Sur la convergence en moyenne, C. R., 144 (1907), 1022—4.

Gabriel, R. M. [1] The rearrangement of Fourier positive coefficients, P. L. M. S., 33 (1931), 32—51.

Gergen, J. J. [1] Convergence and summability criteria for Fourier series, Q. J. (Oxford series), 1 (1930), 252—275.

Gronwall, T. H. [1] On the Fourier coefficients of a continuous function, B. A. M. S., 27 (1921), 320—321. — [2] Zur Gibbs'schen Erscheinung, Ann. M., 3 (1931), 233—240. — [3] Über einige Summationsmethoden und ihre Anwendung auf die Fouriersche Reihe, J. f. M., 147 (1916), 16—35.

Gross, W. [1] Zur Poissonschen Summierung, W. B., 124 (1915), 1017—1037.

Monatshefte für Math.,
Hahn, H. [1] Über Folgen linearer Operationen,
32 (1922), 3—88. — [2] Über Fejér's Summierung der Fourierschen Reihen,
J. D. M. V., 25 (1917), 359—366.

Taylor's series, Q. J., 44 (1913).
Hardy, G. H. [1] A theorem concerning
147—160. — [2] On the summability of Fourier series, P. L. M. S., 12 (1913),
365—372. — [3] On certain criteria for the convergence of the Fourier series of
a continuous function, M. M., 47 (1918), 149—156.—[4] Weierstrass's non-differ-
entiable function, T. A. M. S., 17 (1916), 301—325. — [5] Theorems relating to
summability and convergence of slowly oscillating series, P. L. M. S., 8 (1910),
301—320. — [6] The summability of a Fourier series by logarithmic means.
Q. J. (Oxford series), 2 (1931), 107—112.—[7] Notes on some points in the inte-
gral calculus, LV, M. M., 51 (1922), 186—192. — [8] A new proof of the function-
al equation of the Zeta-function, Matematisk Tidskrift, B., 1922, 71—73. —
[9] Remarks on three recent notes in the Journal. J. L. M. S. 3 (1928), 166—169.—
[10] The mean value of the modulus of an analytic function, P. L. M. S., 14
(1914), 269—277.

On Young's convergence criterion
Hardy, G. H. and Littlewood, J. E. [1]
for Fourier series, P. L. M. S., 28 (1928), 301—311. — [2] Some new convergence
criteria for Fourier series, J. L. M. S., 7 (1932), 252—256. — [3] Some new
convergence criteria for Fourier series, Annali Scuola Norm. Super. Pisa, 3
(1932), 43—46. — [4] The allied series of a Fourier series, P. L. M. S., 24 (1925),
211—246. — [5] The Fourier series of a positive function, J. L. M. S., 1 (1926),
134—138. — [6] Some properties of fractional integrals, I, M. Z. 27 (1928), 565—606;
II, M. Z., 34 (1932), 403—439. — [7] Solution of the Cesàro summability pro-
blem for power series and Fourier series, M. Z., 19 (1923), 67—96. — [8] On the
absolute convergence of Fourier series, J. L. M. S., 3 (1928), 250—253. — [9] Some

problems of Diophantine approximation: A remarkable trigonometrical series, Proc. National Acad. Sc., 2 (1916), 583—586. — [10] Some new properties of Fourier constants, M. A. 97 (1926), 159—209. — [11] Theorems concerning Cesàro means of power series, P. L. M. S., 36 (1934), 516—531. — [12] Some properties of conjugate functions, J. f. M., 167 (1932), 405—423. — [13] A convergence criterion for Fourier series, M. Z., 28 (1928), 612—634. — [14] On Parseval's theorem, P. L. M. S., 26 (1927), 287—294. — [15] Some new properties of Fourier constants, J. L. M. S., 6 (1931), 3—9. — [16] Sur la série de Fourier d'une fonction à carré sommable, C. R., 156 (1913), 1307—1309. — [17] A maximal theorem with function-theoretic applications, A. M., 54 (1930), 81—116. — [18] Some problems of Diophantine approximation: The trigonometrical series associated with the elliptic Theta-function, A. M., 37 (1914), 193—238. — [19] On the series conjugate to the Fourier series of a bounded function, J. L. M. S., 6 (1931), 278—286. — [20] Two theorems concerning Fourier series, J. L. M. S., 1 (1926), 19—25.

Hausdorff, F. [1] Summationsmethoden und Momentfolgen, M. Z., 9 (1921), 74—109, 280—299. — [2] Eine Ausdehnung des Parsevalschen Satzes über Fourierreihen, M. Z., 16 (1923), 163—169.

Helly, E. [1] Über lineare Funktionaloperationen, W. B., 121 (1912), 265—297.

Hille, E. [1] Note on a power series considered by Hardy and Littlewood, J. L. M. S., 4 (1929), 176—182.

Hille, E. and **Tamarkin, J. D.** [1] On the summability of Fourier series, I, T. A. M. S., 34 (1932), 757—783; II, Ann. M., 34 (1933), 329—344, 602—605; III, M. A., 108 (1933), 525—577. — [2] Remarks on a known example of a monotone continuous function, American Math. Monthly, 36 (1929), 255—264. — [3] On the theory of Fourier transforms, B. A. M. S., 39 (1933), 768—774.

Hobson, E. W. [1] On the uniform convergence of Fourier series, P. L. M. S., 5 (1907), 275—289. — [2] On the integration of trigonometrical series, J. L. M. S., 2 (1927), 164—166. — [3] On a general convergence theorem, P. L. M. S., 6 (1908), 349—395.

Jackson, D. [1] Über eine trigonometrische Summe, R. P., 32 (1911), 257—262.

Jacob, M. [1] Summierbarkeit von Fourierschen Reihen und Integralen, M. Z., 29 (1929), 20—33.

Jensen, J. L. W. V. [1] Sur les fonctions convexes et les inégalités entre les valeurs moyennes, A. M., 30 (1906), 175—193.

Jerosch, F. and Weyl, H. [1] Über die Konvergenz von Reihen die nach periodischen Funktionen fortschreiten, M. A., 64 (1909), 67—80.

Kaczmarz, S. [1] Über ein Orthogonalsystem, C. Rendus du I Congrès des mathématiciens des pays slaves, Warsaw, 1929, 189—192. — [2] Sur la convergence et la sommabilité des développements orthogonaux, S. M., 1 (1929), 87—121. — [3] Integrale vom Dinischen Typus, S. M., 3 (1931), 189—199. — [4] The divergence of certain integrals, J. L. M. S., 7 (1932), 218—222. — [5] On some classes of Fourier series, J. L. M. S., 8 (1933), 39—46.

Kaczmarz, S. and **Steinhaus, H.** [1] Le système orthogonal de M. Rademacher, S. M., 2 (1930), 231—247.

Khintchine, J. [1] Über dyadische Brüche, M. Z., 18 (1923), 109—116.

Khintchine, J. und **Kolmogoroff, A.** [1] Über Konvergenz von Reihen deren Glieder durch den Zufall bestimmt werden, Recueil de la Soc. Math. de Moscou, 32 (1925), 668—677.

Kogbetliantz, E. [1] Analogies entre les séries trigonométriques et les séries sphériques, A. E. N., 40 (1923), 259—323. — [2] Recherches sur l'unicité des séries ultrasphériques, J. M., 5 (1924), 125—196.

Kolmogoroff, A. [1] Sur l'ordre de grandeur des coefficients de la série de Fourier-Lebesgue, B. A. P., 1923, 83—86. — [2] Sur la possibilité de la définition générale de la dérivée, de l'intégrale et de la sommation des séries divergentés, C. R., 180 (1925), 365—364. — [3] Über die Summen durch den Zufall bestimmten unabhängigen Grössen, M. A., 99 (1928), 309—319. — [4] Sur les fonctions harmoniques conjuguées et les séries de Fourier, F. M., 7 (1925), 23—28. — [5] Sur un procédé d'intégration de M. Denjoy, F. M., 11 (1928), 27—28. — [6] Une série de Fourier-Lebesgue divergente partout, C. R., 183 (1926), 1327—8. — [7] Une série de Fourier-Lebesgue divergente presque partout, F. M., 4 (1923), 324—328. — [8] Une contribution à l'étude de la convergence des séries de Fourier, F. M., 5 (1924), 96—97.

Kolmogoroff, A. and Khintchine, J., see **Khintchine, J. and Kolmogoroff, A.**

Kolmogoroff, A. and **Seliverstoff, G.** [1] Sur la convergence des series de Fourier, C. R., 178 (1925), 303—305. — [2] Sur la convergence des séries de Fourier, A. L., 3 (1926), 307—310.

Landau, E. [1] Über eine trigonometrische Ungleichung, M. Z., 37 (1933), 36.

Lebesgue, H. [1] Sur la représentation trigonométrique approchée des fonctions satisfaisant à une condition de Lipschitz, B. S. M. F., 38 (1910), 184—210. — [2] Sur les intégrales singulières, Annales de Toulouse, I (1909), 25—117. — [3] Recherches sur la convergence des séries de Fourier, M. A., 61 (1905), 251—280.

Lévy, P. [1] Sur la convergence absolue des séries de Fourier, Compositio Math., 1 (1934), 1—14.

Liouville, J. [1] Sur le calcul des différentielles à indices quelconques, Journal de l'Ecole Polytéchn., 13 (1832), 1—69.

Littlewood, J. E. [1] On the mean values of power series, P. L. M. S., 25 (1924), 328—337. — [2] On mean values of power series, J. L. M. S., 5 (1930), 179—182. — [3] On a theorem of Kolmogoroff, J. L. M. S., 1 (1926), 229—231. — [4] On a theorem of Zygmund, J. L. M. S., 4 (1929), 305—307.

Littlewood, J. E. and Hardy, G. H., see Hardy, G. H. and **Littlewood, J. E.**

Littlewood, J. E. and **Paley, R. E. A. C.** [1] Theorems on Fourier series and power series, J. L. M. S., 6 (1931), 230—233.

Lukàcs, F. [1] Über die Bestimmung des Sprunges einer Funktion aus ihrer Fourierreihe, J. f. M., 150 (1920), 107—112.

Lusin, N. [1] Integral and trigonometrical series (in Russian), Moscow, 1915, 1—242. — [2] Sur la notion de l'intégrale, Annali di Mat., 26 (1916), 77—129. — [3] Sur l'absolue convergence des séries trigonométriques, C. R., 155 (1912), 580—582.

Marcinkiewicz, J. [1] A new proof of a theorem on Fourier series, J. L. M. S., 8 (1933), 279. — [2] Sur deux méthodes de sommation de Riemann, J. L. M. S. (to be published).

Mazurkiewicz, S. [1] Sur les séries de puissances, F. M., 3 (1922), 52—58.

Menchoff, D. [1] Sur l'unicité du développement trigonométrique, C. R., 163 (1916), 433—436.

Mirimanoff, D. [1] Remarque sur la notion d'ensemble parfait de 1-re espèce, F. M., 4 (1923), 122—123.

Moore, C. N. [1] On the application of Borel's method to the summation of Fourier series, Proc. Nat. Acad. Sc., 11 (1925), 284—287.

Mulholland, H. P. [1] Concerning the generalization of the Young-Hausdorff theorem, P. L. M. S., 35 (1933), 257—293.

Nalli, P. [1] Sulle derivate seconde generalizzate, Giornale di Matematiche, 53 (1915), 169—177.

Neder, A. [1] Konvergenzdefekte der Potenzreihen stetiger Funktionen auf dem Rande des Konvergenzkreises, M. Z., 6 (1920), 262—269. — [2] Zur Theorie der trigonometrischen Reihen. M. A., 84 (1921), 117—136.

Niemytzki, V. [1] Sur quelques classes d'ensembles linéaires avec applications aux séries trigonométriques absolument convergentes, Rec. de la Soc. Math. de Moscou, 33 (1926), 5—32 (in Russian, with a résumé in French).

Orlicz, W. [1] Über eine gewisse Klasse von Räumen vom Typus B, B. A. P., 1932, 207—220.

Orlicz, W. and Birnbaum, Z., see **Birnbaum, Z. and Orlicz, W.**

Paley, R. E. A. C. [1] A remarkable system of orthogonal functions, P. L. M. S., 34 (1932), 241—279. — [2] A proof of a theorem on bilinear forms, J. L. M. S., 6 (1931), 226—230. — [3] A note on power series, J. L. M. S., 7 (1932), 122—130. — [4] Some theorems on orthogonal functions, S. M., 3 (1931), 226—238. — [5] On the lacunary coefficients of power series, Ann. M., 34 (1933), 615—616. — [6] A proof of a theorem on averages, P. L. M. S., 31 (1930), 289—300. — [7] On Fourier series with positive coefficients, P. L. M. S., 7 (1932), 205—208.

Paley, R. E. A. C. and Zygmund, A. [1] On some series of functions, Proc. of the Cambridge Phil. Soc., 26 (1930), 337—357, 458—474, and 28 (1932), 190—205. — [2] On the partial sums of Fourier series, S. M., 2 (1930), 221—227.

Plancherel, M. [1] Contribution à l'étude de la représentation d'une fonction arbitraire par des intégrales définies, R. P., 30 (1910), 289—335. — [2] Sur la convergence et la sommabilité par les moyennes de Cesàro de $\lim_{z=\infty}\int_a^z f(x)\cos xy\,dx$, M. A., 76 (1915), 315—326.

Paley, R. E. A. C. and Littlewood, J. E., see **Littlewood, J. E. and Paley, R. E. A. C.**

Plessner, A. [1] Eine Kennzeichnung der totalstetigen Funktionen, J. f. M., 160 (1929), 26—32. — [2] Zur Theorie der konjugierten trigonometrischen Reihen, Mitteilungen des Math. Seminar d. Univ. Giessen, 10 (1923), 1—36. — [3] Über das Verhalten analytischer Funktionen auf dem Rande des Definitionsbereiches, J. f. M., 158 (1927), 219—227. — [4] Über Konvergenz von trigonometrischen Reihen, J. f. M., 155 (1925), 15—25.

Pollard, S. [1] On the criteria for the convergence of a Fourier series, J. L. M. S., 2 (1927), 255—262.

Prasad, B. N. [1] Contribution à l'étude d'une série conjuguée à une série de Fourier, J. M., 9 (1932), 153—205. — [2] On the summability of Fourier series and the bounded variation of power series, P. L. M. S., 36 (1933), 407—424. — [3] A theorem on the summability of the allied series of a Fourier series, J. L. M. S., 6 (1931), 274—278.

Pringsheim, A. [1] Über das Verhalten von Potenzreihen auf dem Convergenzkreise, Münchener Sitzungsb., 30 (1900), 37—100. — [2] Über neue Gültigkeitsbedingungen für die Fouriersche Integralformel, M. A., 68 (1910), 367—408.

Privaloff, J. [1] Sur la dérivation des séries de Fourier, R. P., 41 (1916) 202—206. — [2] Intégrale de Cauchy, Saratoff, 1919, p. 1—94 (in Russian). — [3] Sur les fonctions conjuguées, B. S. F. M. 44 (1916), 100—103. — [4] Sur la convergence des séries trigonométriques conjuguées, Rec. de la Soc. Math. de Moscou, 32 (1925), 357—363.

Rademacher, H. [1] Einige Sätze über Reihen von allgemeinen Orthogonalfunktionen, M. A., 87 (1922), 112—138.

Rajchman, A. [1] Sur l'unicité du développement trigonométrique, F. M. 3 (1922), 286—302. — [2] Sur le principe de localisation de Riemann, Comptes Rendus de la Soc. Scient. de Varsovie, 11 (1918), 115—122 (in Polish, with a résumé in French). — [3] Sur la multiplication des séries trigonométriques et sur une classe d'ensembles fermés, M. A. 95 (1926), 388—408. — [4] Une classe des séries trigonométriques qui convergent presque partout vers zéro, M. A., 101 (1929), 686—700. — [5] Séries trigonométriques sommables par le procédé de Poisson, Prace Matematyczno-Fizyczne, 30 (1919), 19—88 (in Polish, with a résumé in French).

Rajchman and Zygmund, [1] Sur la possibilité d'appliquer la méthode de Riemann aux séries trigonométriques sommables par le procédé de Poisson,

M. Z., 25 (1926), 261—273. — [2] Sur la relation du procédé de sommation de Cesàro et celui de Riemann, B. A. P., 1925, 69—80.

Riemann, B. [1] Über die Darstellbarkeit einer Funktion durch eine trigonometrische Reihe, Ges. Werke, 2. Aufl., Leipzig 1892, p. 227—271.—[2] Versuch einer allgemeinen Auffassung der Integration und Differentiation, Ges. Werke, p. 331—344.

Riesz, F. [1] Über orthogonale Funktionensysteme, Göttinger Nachrichten, 1907, 116—122. — [2] Untersuchungen über Systeme integrierbarer Funktionen, M. A., 69 (19 0), 449—497. — [3] Sur les polynomes trigonométriques, C. R., 158 (1914), 1657—1661.—[4] Über die Randwerte einer analytischen Funktion, M. Z., 18 (1922), 87—95. — [5] Über die Fourierkoeffizienten einer stetigen Funktion von beschränkter Schwankung, M. Z., 2 (1918), 312—315.—[6] Über eine Verallgemeinerung des Parsevalschen Formel, M. Z., 18 (1923), 117—124.—[7] Sur un théorème de maximum de M. M. Hardy et Littlewood, J. L. M. S., 7 (1932), 10—13.—[8] Sur une inégalité intégrale, J. L. M. S., 5 (1930), 162—168.—[9] Sur la formule d'inversion de Fourier, A. S., 3 (1927), 235—241. — [10] Sur certains systèmes singuliers d'équations intégrales, A. E. N., 28 (1911), 33—62.

Riesz, F. and M. [1] Über Randwerte einer analytischen Funktion, Quatrième Congrès des math. scandinaves, 1916, 27—44.

Riesz, M. [1] Sur les séries de Dirichlet et les séries entières, C. R., 149 (1909), 309—312. — [2] Sur la sommation des séries de Fourier, A. S., 1 (1923), 104—113.—[3] Sur les maxima des formes billinéaires et sur les fonctionnelles linéaires, A. M., 49 (1926), 465—497. — [4] Sur les fonctions conjuguées, M. Z., 27 (1927), 218—244. — [5] Neuer Beweis des Fatouschen Satzes, G. N., 1916, 12—65. — [6] Sätze über Potenzreihen, A. f. M., 11 (1916), No 12. — [7] Über summierbare trigonometrische Reihen, M. A., 71 (1911), 54—75.

Rogosinski, W. [1] Über positive harmonische Entwickelungen und typisch reelle Potenzreihen, M. Z., 35 (1932), 93—121.—[3] Über den Einfluss einseitiger Eigenschaften einer Funktion auf ihre Fourierreihe, Schriften der Königsberger Gelehrten Ges., 1926, Heft 3.—[3] Über die Abschnitte trigonometrischer Reihen, M. A., 95 (1925), 110—134.—[4] Reihensummierung durch Abschnitts-Koppelungen, M. Z., 25 (1926), 132—149.

Saks, S. [1] On some functionals, T. A. M. S., 35 (1933), 549—556.

Salem, R. [1] Detérmination de l'ordre de grandeur à l'origine de certaines séries trigonométriques, C. R., 186 (1928), 1804—1806. — [2] Sur une propriété des séries de Fourier des fonctions de carré sommable, C. R., 197 (1933), 113—115.

Seliverstoff, G. and **Kolmogoroff, A.**, see **Kolmogoroff, A.** and **Seliverstoff, G.**

Smirnoff, V. [1] Sur les valeurs limites des fonctions analytiques, C. R., 188 (1929), 131—133.

Stein, P. [1] On a theorem of M. Riesz, J. L. M. S., 8 (1933), 242—247.

Steinhaus, H. [1] Sur le développement du produit de deux fonctions en une série de Fourier, Bulletin Intern. de l'Acad. de Cracovie, 1913, 113—116.— [2] Sur quelques propriétés des séries trigonométriques et de celles de Fourier, Rozprawy Akademji Umiejętności, Cracow, 56 (1925), 175—225 (in Polish, with a résumé in French).—[3] Additive und stetige Funktionaloperationen, M. Z., 5 (1919), 186—221. — [4] A new property of G. Cantor's set, Wektor, 7 (1917) (in Polish). — [5] Sur les distances des points des ensembles de mesure positive, F. M., 1 (1920), 93—104. — [6] Sur la convergence non uniforme des séries de Fourier, Bull. de l'Acad. de Cracovie, 1913, 145—160. — [7] Sur les défauts de convergence des séries trigonométriques et potentielles, B. A P., 1919, 123—141.— [8] A generalization of G. Cantor's theorem on trigonometrical series, Wiadomości Matematyczne, 24 (1920), 197—201 (in Polish). — [9] Une série trigonométrique partout divergente, Comptes Rendus de la Soc. Scient. de Varsovie, 1912, 219—229. — [10] A divergent trigonometrical series, J. L. M. S., 4 (1929), 86—88.

Steinhaus, H. and Kaczmarz, S., see **Kaczmarz, S.** and **Steinhaus, H.**

Szász, O. [1] Über die arithmetische Mittel Fourierscher Reihen, A. M., 48 (1926), 355—362.—[2] Über den Kongvergenzexponent der Fourierschen Reihen, Münchener Sitzungsberichte, 1922, 135—150.—[3] Über die Fourierschen Reihen gewisser Funktionenklassen, M. A. 100 (1928), 530—536.

Szegö, G. [1] Entwickelung einer willkürlichen Funktion nach den Polynomen eines Orthogonalsystems, M. Z. 12 (1922), 61—94. — [2] Über die Lebesgueschen Konstanten bei den Fourierreihen, M. Z., 9 (1921), 163—166.

Szidon, S. [1] Reihentheoretische Sätze und ihre Anwendungen in der Theorie der Fourierschen Reihen, M. Z., 10 (1921), 121—127.—[2] Verallgemeinerung eines Satzes über die absolute Konvergenz von Fourierreihen mit Lücken, M. A., 97 (1927), 675—676.—[3] Einige Sätze und Fragestellungen über Fourierkoeffizienten, M. Z., 34 (1932), 477 — 480. — [4] Ein Satz über trigonometrische Polynome und seine Anwendung in der Theorie der Fourierreihen, M. A., 106 (1932), 536—539.

Tamarkin, J. D. [1] Remarks on the theory of conjugate functions, P. L. M. S., 34 (1932), 379—391.

Tamarkin, J. D. and Hille, E. see **Hille, H. and Tamarkin, J. D.**

Titchmarsh, E. C. [1] The order of magnitude of the coefficients in a generalized Fourier series, P. L. M. S., 22 (1925), xxv—xxvi. — [2] The convergence of certain integrals, P. L. M. S., 24 (1925), 347—358. — [3] Additional note on conjugate functions, P. L. M. S., 4 (1929), 204 — 206. — [4] On conjugate functions, P. L. M. S., 29 (1929), 49—80.—[5] Reciprocal formulae for series and integrals, M. Z., 25 (1926), 321—347.— [6] A contribution to the theory of Fourier transforms, P. L. M. S., 23 (1924), 279—289.

Toepler, A. [1] Bemerkenswerthe Eigenschaften der periodischen Reihen, Wiener Akad. Anz., 13 (1876), 205—209.

Toeplitz, O. [1] Über allgemeine lineare Mittelbildungen, Prace Matematyczno-Fizyczne, 22 (1911), 113—119.

Tonelli, L. [1] Sulla convergenza delle serie di Fourier, A. L., II (1925), 85—91. — [2] Sulla convergenza assoluta delle serie di Fourier, A. L., II (1925), 145—149.

de la Vallée-Poussin, Ch. J. [1] Un nouveau cas de convergence des séries de Fourier, R. P., 31 (1911), 296—299. - [2] Sur l'intégrale de Lebesgue, T. A. M. S., 16 (1915), 435 — 501. — [3] Sur l'unicité du développement trigonométrique, Bull. de l'Acad. Royale de Belgique, 1912, 702 — 718. — [4] Sur l'approximation des fonctions d'une variable réelle, Bull. de l'Acad. royale de Belgique, 1908, 193—254.

Verblunsky, S. [1] On some classes of Fourier series, P. L. M. S., 33 (1932), 287—327.—[2] Some theorems on generalized Fourier constants, P. L. M. S., 37 (1934), 33 — 62. — [3] On the theory of trigonometric series, I, P. L. M. S., 34 (1932), 441 — 456; II, P. L. M. S., 34 (1932), 457 — 491; III, P. L. M. S., 34 (1932), 526—560; IV, P. L. M. S., 35 (1933), 445—487; V, F. M., 21 (1933), 168—210.— [4] Note on summable trigonometric series, J. L. M. S., 6 (1931), 106 — 112. — [5] The relation between Riemann's method of summation and Cesàro's, Proc. Cambridge Phil. Soc., 26 (1930), 34—42.

Walsh, J. L. [1] A closed set of normal orthogonal functions, American Journal of Math., 55 (1923), 5—24.

Waraszkiewicz, Z. [1] Sur un théorème de M. Zygmund, B. A. P., 1929, 275—279.

Warschawski, S. [1] Über einige Konvergenzsätze aus der Theorie der konformen Abbildung, Göttinger Nachr., 1930, 344—359.

Watson, G. N. [1] General transforms, P. L. M. S., 35 (1933), 156—199.

Weyl, H. [1] Bemerkungen zum Begriff der Differentialquotienten gebrochener Ordnung, Vierteljahrsschrift der Naturf. Ges. in Zürich, 62 (1917), 296—302.

Weyl, H. and **Jerosch, F.**, see **Jerosch, F.** and **Weyl, H.**

Whittaker, J. M. [1] The absolute summability of Fourier series, Proc. Edinburgh Math. Soc., 2 (1930), 1—5.

Wiener, N. [1] Tauberian theorems, Ann. M, 33 (1932), 1—100. — [2] The quadratic variation of a function and its Fourier coefficients, Massachusett's Journal of Math., 3 (1924), 72—94.

Wilton, J. R. [1] An approximate functional equation of a simple type: Applications to certain trigonometrical series, J. L. M. S., 9 (1934), 247—254.

Young, W. H. [1] On Fourier series and functions of bounded variation, Proc. Roy. Soc. (A), 88 (1913), 561—568. — [2] Konvergenzbedingungen für die verwandte Reihe einer Fourierschen Reihe, Münchener Sitzungsb., 41 (1911), 361—371. — [3] On the mode of oscillation of a Fourier series and of its allied series, P. L. M. S., 12 (1913), 433—452. — [4] On the convergence of the derived series of Fourier series, P. L. M. S., 17 (1916), 195—236. — [5] On the convergence of a Fourier series and of its allied series, P. L. M. S., 10 (1912), 254—272. — [6] The usual convergence of a class of trigonometrical series, P. L. M. S., 13 (1914), 13—28. — [7] On classes of summable functions and their Fourier series, Proc. Roy. Soc., (A) 87 (1912), 225—229. — [8] On the condition that a trigonometrical series should have a certain form, Proc. London Roy. Soc., (A) 88 (1913), 569—574. — [9] On the Fourier series of bounded functions, P. L. M. S., 12 (1913), 41—70. — [10] On successions with subsequences converging to an integral, P. L. M. S., 24 (1926), 1—20. — [11] On the integration of Fourier series, P. L. M. S., 9 (1911), 449—462. — [12] On the multiplication of successions of Fourier constants, Proc. Roy. Soc. (A) 87 (1912), 331—339. — [13] On the determination of the summability of a function by means of its Fourier coefficients, P. L. M. S., 12 (1913), 71—88. — [14] A note on trigonometrical series, M. M., 38 (1909), 44—48. — [15] On the ordinary convergence of restricted Fourier series, Proc. Roy. Soc. (A), 93 (1917), 276—292. — [16] On restricted Fourier series and the convergence of power series, P. L. M. S., 17 (1918), 353—366.

Young, W. H. and **G. C.** [1] On the theorem of Riesz-Fischer, Q. J., 44 (1913), 49—88.

Zalcwasser, Z. [1] Sur le phénomène de Gibbs dans la théorie des séries de Fourier, F. M., 12 (1928), 126—151.

Zygmund, A. [1] Sur la dérivation des séries de Fourier, B. A. P., 1924, 243—249. — [2] Sur la sommation des séries trigonométriques conjuguées aux séries de Fourier, B. A. P., 1924, 251—258. — [3] Sur un théorème de M. Fekete, B. A. P., 1927, 343—347. — [4] Sur les fonctions conjuguées, F. M., 13 (1929), 284—303, Corrigenda, F. M., 18 (1932), 312. — [5] On the convergence of lacunary trigonometric series, F. M., 16 (1930), 90—107, Corrigenda, F. M., 18 (1932), 312. — [6] Quelques théorèmes sur les séries trigonométriques et celles de puissances, S. M., 3 (1931), 77—91. — [7] Some points in the theory of trigonometric and power series, T. A. M. S., 36 (1934), 586—617. — [8] Sur la convergence absolue des séries de Fourier, P. L. M. S., 3 (1928) 194—196. — [9] A remark on conjugate series, P. L. M. S., 34 (1932), 392—400. — [10] Sur l'application de la première moyenne arithmétique, F. M., 10 (1926), 356—362. — [11] Sur la théorie riemannienne des séries trigonométriques, M. Z., 24 (1926), 47—104. — [12] Contribution à l'unicité du développement trigonométrique, M. Z., 24 (1926), 40—46. — [13] On lacunary trigonometric series, T. A. M. S., 34 (1932), 435—446. — [14] Sur les séries trigonométriques sommables par le procédé de Poisson, M. Z., 25 (1926), 274—290. — [15] Sur un théorème de M. Gronwall, B. A. P., 1925, 207—217.

Zygmund, A. and **Paley, R. E. A. C.**, see **Paley, R. E. A. C.** and **Zygmund, A.**

Zygmund, A. and **Rajchman, A.**, see **Rajchman, A.** and **Zygmund, A.**